TRACE-CLASS NORM MULTIPLIERS

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ABSTRACT. In this paper we study the pointwise multipliers of the space $L^2(X, \nu) \hat{\otimes} L^2(Y, \mu)$ for different measure spaces (X, ν) and (Y, μ) . First we consider the case where X and Y are finite sets, then we use this to study the multipliers of $L^2(I, \nu) \hat{\otimes} L^2(I, \nu)$, where I is the unit interval and ν any Borel measure on I .

0. Introduction. Let A be a Banach space of complex-valued functions on a measure space (X, ν) such that A vanishes at no point of X . A multiplier of A is a complex-valued function φ on X such that $\varphi f \in A$ for all $f \in A$, where $(\varphi f)(x) = \varphi(x)f(x)$. Clearly, $M(A)$, the set of all multipliers of A , forms an algebra which will be called the multiplier algebra of A .

Multiplier algebras have been studied by Schur, [8], for the space of matrices describing bounded linear operators on l^2 , and by Bennett, [2], for certain other spaces of matrices. Bennett proved the result: a matrix $F: l^p \rightarrow l^q$ is a multiplier of $L(l^p, l^q)$ (the space of bounded linear operators from l^p into l^q) if and only if, for each $d \in l^p$, the matrix $F \circ (\text{diag } d): l^p \rightarrow l^q$ is (q, q) -absolutely summing. The space $L^2(X, \nu) \hat{\otimes} L^2(X, \nu)$, the completion of the projective tensor product with itself [1], can be realized as a space of functions on $(X \times X, \nu \otimes \nu)$. The author [5] showed that $\mathfrak{N}(l^2 \hat{\otimes} l^2)$ is isomorphic to the space of all functions φ on $N \times N$ (N is the set of natural numbers) such that there exists a sequence $(\varphi_n)_{n=1}^\infty$ uniformly bounded in $l^\infty \hat{\otimes} l^\infty$ which converges to φ pointwise. The previous result of the author is included implicitly in that of Bennett [2].

The object of this paper is to study the multiplier algebra of $L^2(X, \nu) \hat{\otimes} L^2(X, \nu)$ for a nondiscrete space X . In particular we study the multiplier algebra of $L^2(I, \nu) \hat{\otimes} L^2(I, \nu)$, where I is the unit interval and ν is any Borel measure on I . As is known [7] $L^2(X, \nu) \hat{\otimes} L^2(X, \nu)$ is isometrically isomorphic to the space of trace-class operators on $L^2(X, \nu)$. This implies that if $\varphi \in \mathfrak{N}(L^2(I, \nu) \hat{\otimes} L^2(I, \nu))$,

$$\|\varphi\|_M = \sup |\langle \varphi \cdot u \otimes v, U \rangle|,$$

where the supremum is taken over all atoms $u \otimes v$ and all operators U in the unit balls of $L^2(X, \nu) \hat{\otimes} L^2(X, \nu)$ and $L(L^2(X, \nu))$ respectively.

Throughout this paper, H denotes a real (or complex) Hilbert space of dimension n . If A is a Banach space then A^* denotes its dual and $L(A)$ is the space of bounded linear operators on A . An atom in $A \hat{\otimes} A$ is an element of the form

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$f \otimes g$, f and g are in A . For $a \in A$, $\|a\|_A$ denotes the norm of a in A . The space of p -summable sequences will be denoted by l^p , and l^∞ stands for the space of bounded sequences. Finally, l_n^2 will denote the n -dimensional space of square summable sequences, Z_n^+ stands for the set $\{1, 2, \dots, n\}$, and p^* is the conjugate of p ($1/p + 1/p^* = 1$).

1. Multipliers and normal contractions. We denote by H a complex Hilbert space. The set $D = \{Z \in \mathbb{C} \mid |Z| \leq 1\}$ will denote the unit disc, and $C(D)$ the space of continuous functions on D . For any pair of normal contractions S and T on H and for any element φ in $C(D) \hat{\otimes} C(D)$, one can make sense of $\varphi(S, T)$. Let $E_1 = \{\lambda_1, \dots, \lambda_n\}$ and $E_2 = \{\eta_1, \dots, \eta_n\}$ be any two sets in D , where n is the dimension of H . If Z_n^+ is the set $\{1, \dots, n\}$, then consider the function $\psi: Z_n^+ \times Z_n^+ \rightarrow \mathbb{C}$ defined by

$$\psi(i, j) = \varphi(\lambda_i, \eta_j), \quad 1 \leq i, j \leq n.$$

Then we prove the following.

THEOREM 1.1. *The following are equivalent.*

- (i) $\psi \in M(l_n^2 \hat{\otimes} l_n^2)$ and $\|\psi\|_M < 1$.
- (ii) $\|\varphi(S, T)\|_{L(H)} < 1$ for any pair of normal (not necessarily commuting) contractions S and T on H whose eigenvalues are the elements of the sets E_1 and E_2 respectively.

PROOF. (i) $\rightarrow$ (ii). Let S and T be any pair of normal contractions on H with E_1 and E_2 their sets of eigenvalues. Let $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ be two sets of orthonormal vectors such that

$$Se_j = \lambda_i e_i \quad \text{and} \quad Tf_j = \eta_j f_j, \quad 1 \leq i, j \leq n.$$

Let $\sum_{k=1}^\infty c_k g_k \otimes h_k$ be a representation of φ in $C(D) \hat{\otimes} C(D)$, and let e and f be any two unit vectors in H . If $e = \sum_{i=1}^n a_i e_i$ and $f = \sum_{j=1}^n b_j f_j$, where $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are unit vectors in l_n^2 , then we have

$$\begin{aligned} |\langle \varphi(S, T)f, e \rangle| &= \left| \left\langle \sum_{k=1}^\infty c_k g_k(S) h_k(T) f, e \right\rangle \right| \\ &= \left| \sum_{i,j=1}^n \varphi(\lambda_i, \eta_j) \bar{a}_i b_j \langle f_j, e_i \rangle \right| \\ &= \left| \sum_{i,j=1}^n \psi(i, j) u(i) v(j) U(j, i) \right|, \end{aligned}$$

where $u(i) = \bar{a}_i$, $v(j) = b_j$, $U(j, i) = \langle f_j, e_i \rangle$. Hence $u \otimes v$ is an atom in $l_n^2 \hat{\otimes} l_n^2$ and U is a unitary operator on l_n^2 . Assumption (i) then implies that $|\langle \varphi(S, T)f, e \rangle| < 1$. This proves (ii).

Conversely (ii) $\rightarrow$ (i). Let $u \otimes v$ be any atom of unit norm in $l_n^2 \hat{\otimes} l_n^2$ and U be any unitary operator on l_n^2 . If $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ are two sets of orthonormal vectors in H such that $U(i, j) = \langle f_i, e_j \rangle$, then define two linear operators S and T on H as follows.

$$S\left(\sum_{i=1}^n a_i e_i\right) = \sum_{i=1}^n a_i \bar{\lambda}_i e_i, \quad T\left(\sum_{j=1}^n b_j f_j\right) = \sum_{j=1}^n b_j \eta_j f_j.$$

Then S and T are two normal contractions on H with sets of eigenvalues E_1 and E_2 respectively. Set $e = \sum_{i=1}^n \bar{u}(i) e_i$, $f = \sum_{j=1}^n v(j) f_j$. Then

$$\begin{aligned} \left| \sum_{i,j=1}^n \psi(i, j) u(i) v(j) U(j, i) \right| &= \left| \sum_{i,j=1}^n \varphi(\lambda_i, \eta_j) u(i) v(j) \langle f_j, e_i \rangle \right| \\ &= |\langle \varphi(S, T) f, e \rangle| < 1 \end{aligned}$$

by assumption (ii). Since the unit ball of $L(l_n^2)$ is just the closed convex hull of the set of unitary operators [6], (i) follows. The proof of the theorem is complete.

Since $M(l_n^2 \hat{\otimes} l_n^2) = l_n^\infty \hat{\otimes} l_n^\infty$, [5], up to norm equivalence, it follows that if φ is a polynomial on $D \times D$ then

$$\|\varphi\|_{C(D) \hat{\otimes} C(D)} \leq C \cdot \|\varphi(S, T)\|_{L(H)},$$

for some constant C independent of φ , for all pairs of normal contractions S and T on H .

2. Multipliers and spectral measures. Let X be a set, F a σ -algebra of subsets of X . A map $P, P: F \rightarrow L(H)$ is called a spectral measure on X if the range of P is contained in the set of projections in $L(H)$, if disjoint sets are taken to orthogonal projections, and if $P(X) = I$. Further the set function $\mu_x: F \rightarrow \mathbb{C}$, $\mu_x(E) = \langle P(E)x, x \rangle$ is a measure on X for all $x \in H$. Now taking X to be the set Z_n^+ and F to be the family of all subsets of Z_n^+ , we prove

THEOREM 2.1. *The following are equivalent.*

- (i) $\varphi \in M(l_n^2 \hat{\otimes} l_n^2)$ and $\|\varphi\|_M \leq 1$.
- (ii) $\|\sum_{i,j=1}^n \varphi(i, j) P(i) Q(j)\|_{L(H)} \leq 1$ for every pair of (not necessarily commuting) spectral measures P, Q on Z_n^+ .

PROOF. (i) $\rightarrow$ (ii). Let P and Q be any two spectral measures on Z_n^+ . Set $P(i)H = A_i$ and $Q(j)H = B_j$, $1 \leq i, j \leq n$. Then

$$H = A_1 \oplus \cdots \oplus A_n, \quad H = B_1 \oplus \cdots \oplus B_n.$$

Let a and b be any two unit vectors in H . If $P(i)a = a_i$ and $Q(j)b = b_j$, then, agreeing not to divide by zero,

$$\begin{aligned} a &= \|a_1\| \cdot a_1 / \|a_1\| + \cdots + \|a_n\| \cdot a_n / \|a_n\|, \\ b &= \|b_1\| \cdot b_1 / \|b_1\| + \cdots + \|b_n\| \cdot b_n / \|b_n\|. \end{aligned}$$

Set $e_i = a_i / \|a_i\|$ if $a_i \neq 0$ and $e_i = 0$ otherwise. Similarly $f_j = b_j / \|b_j\|$ if $b_j \neq 0$ and $f_j = 0$ otherwise. Then $U(i, j) = \langle f_j, e_i \rangle$ is a contraction on l_n^2 . Put $u(i) = \|a_i\|$ and $v(j) = \|b_j\|$, then $u \otimes v$ is an atom of unit norm in $l_n^2 \hat{\otimes} l_n^2$. Now

$$\begin{aligned} \left| \sum_{i,j=1}^n \varphi(i, j) \langle Q(j)b, P(i)a \rangle \right| &= \left| \sum_{i,j=1}^n \varphi(i, j) u(i) v(j) U(j, i) \right| \\ &\leq \|\varphi\|_M \leq 1. \end{aligned}$$

This proves (ii).

Conversely (ii) $\rightarrow$ (i). Let $u \otimes v$ be any atom in the unit ball of $l_n^2 \hat{\otimes} l_n^2$ and U be any unitary operator in $L(l_n^2)$. Choose $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ to be two sets of orthonormal vectors in H such that $U(i, j) = \langle f_i, e_j \rangle$. Put $y = \sum_{i=1}^n \bar{u}(i) e_i$, $x = \sum_{j=1}^n v(j) f_j$, and define $P(i)$ = Projection on the span of $\{e_i\}$ and $Q(j)$ = Projection on the span of $\{f_j\}$. Then we have

$$\left| \sum_{i,j=1}^n \varphi(i, j) u(i) v(j) U(j, i) \right| = \left| \sum_{i,j=1}^n \varphi(i, j) \langle Q(j)x, P(i)y \rangle \right| \\ < \left\| \sum_{i,j=1}^n \varphi(i, j) P(i) Q(j) \right\|_{L(H)} < 1.$$

This completes the proof of the theorem.

Theorem 2.1 can be restated as follows.

THEOREM 2.1'. *The following are equivalent.*

(i) $\varphi \in M(l_n^2 \hat{\otimes} l_n^2)$ and $\|\varphi\|_M < 1$.

(ii) $|\sum_{i,j=1}^n \varphi(i, j) \langle x_i, y_j \rangle| < (\sum_{i=1}^n \|x_i\|^2)^{1/2} (\sum_{j=1}^n \|y_j\|^2)^{1/2}$

for all pairs of sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ of orthogonal elements in H .

One can show the following from Theorem 2.1.

COROLLARY 2.1. (i) $l^1 \hat{\otimes} l^1 \subseteq M(l^2 \hat{\otimes} l^2)$ and $\|\varphi\|_M < K_G \|\varphi\|_{l^1 \hat{\otimes} l^1}$, where K_G is the Grothendieck constant.

(ii) $L(l^2) = (l^2 \hat{\otimes} l^2)^* \subseteq M(l^2 \hat{\otimes} l^2)$ and $\|\varphi\|_M < \|\varphi\|_{L(H)}$.

Now, let $\alpha, \beta: N \rightarrow N$ be any two surjective maps, not necessarily (1-1). Set $\psi = \varphi \circ \alpha \otimes \beta$, and $P'(k) = P\{\alpha^{-1}(k)\}$, $Q'(l) = Q\{\beta^{-1}(l)\}$. Then P' and Q' are spectral projections on N , whenever P and Q are.

A consequence of Theorem 2.1 is the following.

LEMMA 2.1. *If $\varphi \in M(l^2 \hat{\otimes} l^2)$, then $\varphi \circ \alpha \otimes \beta = \psi \in M(l^2 \hat{\otimes} l^2)$. Further, $\|\varphi\|_M = \|\psi\|_M$.*

PROOF. Let $\varphi \in M(l^2 \hat{\otimes} l^2)$ and $\|\varphi\|_M < 1$. Then

$$\left\| \sum_{i,j=1}^n \psi(i, j) P(i) Q(j) \right\|_{L(H)} = \left\| \sum_{k,l} \varphi(k, l) P'(k) Q'(l) \right\|_{L(H)} \\ < \|\varphi\|_M,$$

by Theorem 2.1. Hence $\|\psi\|_M < \|\varphi\|_M$. Similarly, since $\alpha \otimes \beta: N \times N \rightarrow N \times N$ is onto, one can show that $\|\varphi\|_M < \|\psi\|_M$.

Lemma 2.1 implies that if φ is a matrix in $M(l^2 \hat{\otimes} l^2)$, and ψ is another matrix obtained from φ by repeating finitely many rows finitely many times then $\psi \in M(l^2 \hat{\otimes} l^2)$ and $\|\psi\|_M = \|\varphi\|_M$.

Let E be any finite set and μ a discrete measure on E which vanishes at no point of E . We write ν for the counting measure on E . If S is the kernel of an operator in the unit ball of $L(L^2(E, \mu))$, then $\tilde{S}(a, b) = S(a, b)(\mu(a)\mu(b))^{1/2}$ is the kernel of an operator in the unit ball of $L(L^2(E, \nu))$. Similarly if S is the kernel of an operator in

the unit ball of $L(L^2(E, \nu))$, then $\tilde{S}(a, b) = S(a, b) \cdot (\mu(a)\mu(b))^{-1/2}$ is the kernel of an operator in the unit ball of $L(L^2(E, \mu))$. Now if φ is in the unit ball of $M(L^2(E, \mu) \hat{\otimes} L^2(E, \mu))$, then

$$\left| \sum_{a, b \in E} \varphi(a, b) u(a) v(b) S(a, b) \mu(a) \mu(b) \right| < 1$$

for all atoms $u \otimes v$ in the unit ball of $L^2(E, \mu) \hat{\otimes} L^2(E, \mu)$ and all kernels S in the unit ball of $L(L^2(E, \mu))$. Then one can show that $\varphi \in M(L^2(E, \nu) \hat{\otimes} L^2(E, \nu))$ and $\|\varphi\|_M < 1$. Consequently we have the following.

THEOREM 2.2. *Let E be a finite set. Let ν_1 and ν_2 be any two discrete measures on E which are absolutely continuous with respect to each other. Then the following are equivalent.*

- (i) $\varphi \in M(L^2(E, \nu_1) \hat{\otimes} L^2(E, \nu_1))$ and $\|\varphi\|_M < 1$.
- (ii) $\varphi \in M(L^2(E, \nu_2) \hat{\otimes} L^2(E, \nu_2))$ and $\|\varphi\|_M < 1$.

Let E be a finite set and ν a measure on E with atoms $E_1, \dots, E_n$. Set $\tilde{E} = \{a_1, \dots, a_n\}$ with $a_i \in E_i$ and let $\tilde{\nu}$ be the measure on $\tilde{E}$ defined by $\tilde{\nu}(a_i) = \nu(E_i)$. Clearly, if $\tilde{\varphi}(a_i, a_j) = \varphi(E_i, E_j)$, then $\varphi \in M(L^2(E, \nu) \hat{\otimes} L^2(E, \nu))$ and $\|\varphi\|_M < 1$ if and only if $\tilde{\varphi} \in M(L^2(\tilde{E}, \tilde{\nu}) \hat{\otimes} L^2(\tilde{E}, \tilde{\nu}))$ and $\|\tilde{\varphi}\|_M < 1$. Together with Theorem 2.1 and Theorem 2.2 the previous remark implies

THEOREM 2.3. *The following are equivalent.*

- (i) $\varphi \in M(L^2(E, \nu) \hat{\otimes} L^2(E, \nu))$ and $\|\varphi\|_M < 1$.
- (ii) $\|\iint_{E \times E} \varphi(a, b) dP(a) dQ(b)\|_{L(H)} < 1$ for every pair of spectral measures P and Q on (E, ν) .

Now, let X be another set which has the same cardinality as E . Define μ to be a discrete measure on X such that $F: (X, \mu) \rightarrow (E, \nu)$ is a (1-1) onto measure-preserving map. F induces an isometric operator

$$U: L^2(E, \nu) \hat{\otimes} L^2(E, \nu) \rightarrow L^2(X, \mu) \hat{\otimes} L^2(X, \mu),$$

where $U(\varphi) = \varphi \circ F \otimes F$. Then we prove

THEOREM 2.4. *The following are equivalent.*

- (i) $\varphi \in M(L^2(E, \nu) \hat{\otimes} L^2(E, \nu))$ and $\|\varphi\|_M < 1$.
- (ii) $U(\varphi) \in M(L^2(X, \mu) \hat{\otimes} L^2(X, \mu))$ and $\|U(\varphi)\|_M < 1$.

PROOF. (i) $\rightarrow$ (ii). By Theorem 2.2 we can assume without loss of generality that $\nu(E_i) = |E_i|$, the cardinality of E_i . Hence we take μ to be the counting measure on X . Theorem 2.3 together with (i) implies that $\|\iint_{E \times E} \varphi(a, b) dP(a) dQ(b)\|_{L(H)} < 1$, for every pair of spectral measure P and Q on (E, ν) .

Let $u \otimes v$ be any atom in the unit ball of $L^2(X, \mu) \hat{\otimes} L^2(X, \mu)$, and let S be any unitary operator on $L^2(X, \mu)$. Choose two orthonormal sets of vectors in H , $(e_x)_{x \in X}$ and $(f_y)_{y \in X}$, such that $S(x, y) = \langle e_x, f_y \rangle$. Set

$$P(E_i) = \tilde{P}(a_i) = \text{projection onto the span of } \{e_x | F(x) \in E_i\},$$

$$Q(E_j) = \tilde{Q}(a_j) = \text{projection onto the span of } \{f_y | F(y) \in E_j\}.$$

Thus $\|\sum_{i,j=1}^n \tilde{\varphi}(a_i, a_j) \tilde{P}(a_i) \tilde{Q}(a_j)\|_{L(H)} < 1$. Further put $e = \sum_{x \in X} v(x) e_x$ and $f = \sum_{y \in X} \tilde{u}(y) f_y$. Then

$$\begin{aligned} \left| \sum_{x,y} U(\varphi)(x, y) u(x) v(y) S(y, x) \right| &= \left| \sum \tilde{\varphi}(a_i, a_j) \left\langle \sum_{F(x) \in E_i} u(x) e_x, \sum_{F(y) \in E_j} v(y) f_y \right\rangle \right| \\ &= \left| \sum \tilde{\varphi}(a_i, a_j) \langle \tilde{P}(a_i) e, \tilde{Q}(a_j) f \rangle \right| < 1. \end{aligned}$$

This proves (ii).

The proof of (ii) $\rightarrow$ (i) follows from the fact that U is an isometry, and $U(\varphi \cdot \psi) = U(\varphi) \cdot U(\psi)$. This completes the proof of the theorem.

3. Multipliers on infinite compact spaces. Let I denote the unit interval $[0, 1]$, and $\mathcal{F}$ be the σ -algebra of all Borel sets in I . For each positive integer n , let $\mathcal{F}_n$ denote the σ -algebra generated by the 2^n equal length intervals of I whose union is I . If φ is a bounded Borel function on $I \times I$, then $\varepsilon_n(\varphi)$ will denote the conditional expectation of φ with respect to the σ -algebra $\mathcal{F}_n \otimes \mathcal{F}_n$ in $I \times I$. If $\varepsilon_n(\nu)$ is the restriction of ν to $\mathcal{F}_n$, then

$$\int \int_E \varphi(x, y) d\nu(x) d\nu(y) = \int \int_E \varepsilon_n(\varphi)(x, y) d\varepsilon_n(\nu)(x) d\varepsilon_n(\nu)(y),$$

for every set $E \in \mathcal{F}_n \otimes \mathcal{F}_n$. We write $L^2(\nu)$ for $L^2(I, \nu)$, and $\|F\|_{\tau\tau}$ for the norm of F in $L^2(\nu) \hat{\otimes} L^2(\nu)$. Since $L^2(\nu) \hat{\otimes} L^2(\nu)$ is the dual of $L^2(\nu) \check{\otimes} L^2(\nu)$, [6], it follows that $\|F\|_{\tau\tau} = \sup |\langle F, \psi \rangle|$, where the supremum is taken over all compact operators ψ in the unit ball of $L^2(\nu) \check{\otimes} L^2(\nu)$.

THEOREM 3.1. *Let $\varphi, \nu, \mathcal{F}, \varepsilon_n(\varphi), \varepsilon_n(\nu)$ and $\mathcal{F}_n$ be as given above. Then the following are equivalent.*

- (i) $\varphi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\varphi\|_M < 1$.
- (ii) $\varepsilon_n(\varphi) \in M(L^2(I, \mathcal{F}_n, \varepsilon_n(\nu)) \hat{\otimes} L^2(I, \mathcal{F}_n, \varepsilon_n(\nu)))$ and $\|\varepsilon_n(\varphi)\|_M < 1$.

PROOF. (i) $\rightarrow$ (ii). For a start, let $L^2(\varepsilon_n(\nu)) \hat{\otimes} L^2(\varepsilon_n(\nu))$ stand for $L^2(I, \mathcal{F}_n, \varepsilon_n(\nu)) \hat{\otimes} L^2(I, \mathcal{F}_n, \varepsilon_n(\nu))$. Let $f \otimes g$ be an atom in the unit ball of $L^2(\varepsilon_n(\nu)) \hat{\otimes} L^2(\varepsilon_n(\nu))$ and K be the kernel of a contractive compact operator on $L^2(\varepsilon_n(\nu))$. Since $f \otimes g \cdot K \in L^2(\varepsilon_n(\nu) \times \varepsilon_n(\nu))$, it follows, [3], that

$$\varepsilon_n(\varphi \cdot f \otimes g \cdot K) = f \otimes g \cdot K \cdot \varepsilon_n(\varphi).$$

This implies

$$|\langle \varepsilon_n(\varphi) \cdot f \otimes g, K \rangle| = |\langle \varphi \cdot f \otimes g, K \rangle| \leq 1.$$

This proves (ii).

Conversely (ii) $\rightarrow$ (i). If K is the kernel of a contractive compact operator on $L^2(\nu)$, then $\varepsilon_n(K)$ is the kernel of a contractive compact operator on $L^2(\varepsilon_n(\nu))$. Let $f \otimes g$ be an atom in the unit ball of $L^2(\nu) \hat{\otimes} L^2(\nu)$. Then by [3], $\varepsilon_n(f \otimes g)$ is an atom in the unit ball of $L^2(\varepsilon_n(\nu)) \hat{\otimes} L^2(\varepsilon_n(\nu))$. The assumption (ii) implies

$$\left| \int \int_{I \times I} [\varepsilon_n(\varphi) \varepsilon_n(f \otimes g) \cdot \varepsilon_n(K)](x, y) d\varepsilon_n(\nu)(x) d\varepsilon_n(\nu)(y) \right| < 1.$$

Since $\varepsilon_n(\varphi) \cdot \varepsilon_n(f \otimes g) \cdot \varepsilon_n(K) \rightarrow \varphi \cdot f \otimes g \cdot K$ a.e. $\nu \times \nu$, we have

$$\left| \int \int_{I \times I} \varphi(x, y) f(x) g(y) K(x, y) d\nu(x) d\nu(y) \right| \\ = \lim_n \left| \int \int_{I \times I} [\varepsilon_n(\varphi) \varepsilon_n(f \otimes g) \varepsilon_n(K)](x, y) d\varepsilon_n(\nu)(x) d\varepsilon_n(\nu)(y) \right| < 1.$$

This completes the proof.

Let $B(I)$ be the space of all bounded Borel functions on I . $\tilde{B}(I \times I, \nu \times \nu)$ denotes the space of all bounded Borel functions φ on $I \times I$ such that there exists a bounded sequence of functions in $B(I) \hat{\otimes} B(I)$ which converges to φ a.e. $\nu \times \nu$. We then prove

THEOREM 3.2. *For a bounded Borel function φ on $I \times I$, the following are equivalent.*

- (i) $\varphi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$.
- (ii) $\varphi \in \tilde{B}(I \times I, \nu \times \nu)$.

PROOF. (i) $\rightarrow$ (ii). Let $\varphi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\varphi\|_M < 1$. Theorem 3.1 implies that $\varepsilon_n(\varphi) \in M(L^2(\varepsilon_n(\nu)) \hat{\otimes} L^2(\varepsilon_n(\nu)))$ and $\|\varepsilon_n(\varphi)\|_M < 1$. By Theorem 2.2, $\varepsilon_n(\varphi) \in M(L^2(I, \mathcal{F}_n, \mu) \hat{\otimes} L^2(I, \mathcal{F}_n, \mu))$, and $\|\varepsilon_n(\varphi)\|_M < 1$, where μ is the counting measure on $\mathcal{F}_n$. Since $M(l_n^2 \hat{\otimes} l_n^2) = l_n^\infty \hat{\otimes} l_n^\infty$ (up to norm equivalence uniformly in n) [5], we get $\varepsilon_n(\varphi) \in B(I) \hat{\otimes} B(I)$. But $\varepsilon_n(\varphi) \rightarrow \varphi$ a.e. $\nu \times \nu$, it follows that $\varphi \in \tilde{B}(I \times I, \nu \times \nu)$.

Conversely (ii) $\rightarrow$ (i). Let $(\varphi_i)_{i=1}^\infty$ be a sequence of functions in $B(I) \hat{\otimes} B(I)$ such that $\|\varphi_i\|_M < 1$ and $\varphi_i \rightarrow \varphi$ a.e. $\nu \times \nu$. Theorem 3.1 implies that $\varepsilon_n(\varphi_i) \in M(L^2(\varepsilon_n(\nu)) \hat{\otimes} L^2(\varepsilon_n(\nu)))$ and $\|\varepsilon_n(\varphi_i)\|_M < 1$, for all $i > 1$. Theorem 2.2 together with the fact that $M(l_n^2 \hat{\otimes} l_n^2) = l_n^\infty \hat{\otimes} l_n^\infty$ [5] implies that $\varepsilon_n(\varphi_i) \in B(I) \hat{\otimes} B(I)$. Hence $\varepsilon_n(\varphi) \in M(L^2(\varepsilon_n(\nu)) \hat{\otimes} L^2(\varepsilon_n(\nu)))$ and $\|\varepsilon_n(\varphi)\|_M < 1$. By Theorem 3.1, $\varphi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\varphi\|_M < 1$. This completes the proof.

In the final part of the paper we prove a result similar to Theorem 2.3. To start with, let P and Q be any two spectral measures on $(I, \mathcal{F})$, and $\varepsilon_n(P)$ and $\varepsilon_n(Q)$ will denote the restriction of P and Q to $(I, \mathcal{F}_n)$ respectively. If φ is a bounded Borel function on $(I \times I, \mathcal{F} \times \mathcal{F})$, then set

$$S_n = \int \int_{I \times I} \varepsilon_n(\varphi)(x, y) d\varepsilon_n(P)(x) d\varepsilon_n(Q)(y).$$

The sequence $(S_n)_{n=1}^\infty$ need not converge in the weak-operator topology. However if $(S_n)_{n=1}^\infty$ converges weakly, then let $S = \iint_{I \times I} \varphi(x, y) dP(x) dQ(y)$ denote such a limit. One observes that if S exists then it is independent of the sequence $(\mathcal{F}_n)_{n=1}^\infty$. Now we prove

THEOREM 3.3. *For any bounded Borel function φ on $I \times I$ and for any Borel measure ν on I , the following are equivalent.*

- (i) $\varphi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\varphi\|_M < 1$.
- (ii) $\|\iint_{I \times I} \varphi(x, y) dP(x) dQ(y)\|_{L(H)} \leq 1$ for all pairs of spectral measures P and Q on $(I, \mathcal{F})$.

PROOF. (i) $\rightarrow$ (ii). Let $\varphi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\varphi\|_M < 1$. Theorem 3.1 implies that $\varepsilon_n(\varphi) \in M(L^2(\varepsilon_n(\nu)) \hat{\otimes} L^2(\varepsilon_n(\nu)))$ and $\|\varepsilon_n(\varphi)\|_M < 1$ for all $n > 1$. Hence by Theorem 2.3, $\|S_n\|_{L(H)} < 1$ for all $n > 1$. Now we want to prove that $(\langle S_n a, b \rangle)_{n=1}^\infty$ is a convergent sequence of complex numbers for every a and b in H . We show such a sequence is a Cauchy sequence.

If $E = A \times B$ is an atom in $\mathcal{F}_n \times \mathcal{F}_n$, then for $m > n$, $E \in \mathcal{F}_m \times \mathcal{F}_m$. Let $E = \sum_{i,j=1}^k A_i \times B_j$ in $F_m \times F_m$, where $A = \bigcup A_i$, $B = \bigcup B_j$ and $A_i \times B_j$ is an atom in $\mathcal{F}_m \times \mathcal{F}_m$. If $\varepsilon_n(\varphi)(E) = \lambda \in \mathbb{C}$, then

$$\begin{aligned} \varepsilon_n(\varphi)(E) \langle P(A)a, Q(B)b \rangle &= \lambda \langle P(A)a, Q(B)b \rangle \\ &= \sum_{i,j=1}^k \lambda \langle P(A_i)a, Q(B_j)b \rangle. \end{aligned}$$

Hence for each $m > n$, we have a function $\varepsilon_n^m(\varphi)$ which is $\mathcal{F}_m \times \mathcal{F}_m$ measurable and $\varepsilon_n^m(\varphi)(x, y) = \varepsilon_n(\varphi)(x, y)$ for all $(x, y) \in I \times I$. This enables us to write

$$|\langle S_n a, b \rangle - \langle S_m a, b \rangle| = \left| \sum_{r,s} (\varepsilon_n^m(\varphi)(r, s) - \varepsilon_m(\varphi)(r, s)) \cdot \langle Q(E_s)a, P(E_r)b \rangle \right|$$

where $(E_r \times E_s)_{r,s=1}^{2^m}$ are the atoms in $\mathcal{F}_m \times \mathcal{F}_m$. To estimate the last summation, we estimate $\|\varepsilon_n^m(\varphi) - \varepsilon_m(\varphi)\|_M$ as an element of $M(L^2(\varepsilon_m(\nu)) \hat{\otimes} L^2(\varepsilon_m(\nu)))$.

Let $u \otimes v$ be an atom in the unit ball of $L^2(\varepsilon_m(\nu)) \hat{\otimes} L^2(\varepsilon_m(\nu))$ and K a contractive compact operator on $L^2(\varepsilon_m(\nu))$. Since $\varepsilon_m(\varphi \cdot u \otimes v \cdot K) = \varepsilon_m(\varphi) \cdot u \otimes v \cdot K$, and $\varepsilon_n^m(\varphi) \xrightarrow{n,m} \varphi$ a.e. $\nu \times \nu$, we get

$$\langle (\varepsilon_n^m(\varphi) - \varepsilon_m(\varphi))u \otimes v, K \rangle \xrightarrow{n,m} 0.$$

Theorem 2.3 now implies that

$$\lim_{n,m} |\langle S_n a, b \rangle - \langle S_m a, b \rangle| = 0.$$

This proves (ii).

Conversely (ii) $\rightarrow$ (i). The assumption implies that for each $\delta > 0$ there exists an $r(\delta)$ such that $\|S_r\|_{L(H)} < 1 + \delta$. Choose a sequence $\delta_r \rightarrow 0$ as $r \rightarrow \infty$ and $\|S_r\|_{L(H)} < 1 + \delta_r$. By Theorem 2.3 we have

$$\varepsilon_r(\varphi) \in M(L^2(\varepsilon_r(\nu)) \hat{\otimes} L^2(\varepsilon_r(\nu))) \quad \text{and} \quad \|\varepsilon_r(\varphi)\|_M < 1 + \delta_r.$$

For all atoms $u \otimes v$ of unit norm in $L^2(\nu) \hat{\otimes} L^2(\nu)$ and all contractive compact operators K on $L^2(\nu)$ we have

$$\begin{aligned} |\langle \varphi \cdot u \otimes v, K \rangle| &= \lim_r |\langle \varepsilon_r(\varphi) \cdot \varepsilon_r(u \otimes v), \varepsilon_r(K) \rangle| \\ &< \lim_r (1 + \delta_r) < 1. \end{aligned}$$

This completes the proof of the theorem.

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