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SETS WITH FIXED POINT PROPERTY FOR ISOMETRIC MAPPINGS

ANTHONY TO-MING LAU[1]

ABSTRACT. A subset K of a Banach space E is said to have the fixed point property for isometric mappings (f. p. p.) if there exists z in K such that $U(z) = z$ for each isometric mapping U from K onto K. We prove that any bounded closed subsets of E with uniform relative normal structure have the f. p. p. We also prove that if E is either $\mathfrak{B}(H)$ (bounded operators on a Hilbert space H) or $l_\infty(X)$ (bounded functions on X), then E is finite dimensional if and only if each weak*-compact convex subset of E has the f. p. p. This is also equivalent to the convex set of (normal) states on E having the f. p. p.

1. Introduction. A mapping U: $K \to E$ of a set K in a Banach space E into E is called *isometric* if $\|Ux - Uy\| = \|x - y\|$ for all $x, y \in K$. We say that a subset K of E has the *fixed point property for isometric mappings*, or simply f. p. p., if there exists z in K such that $U(z) = z$ for any isometric mapping U of K onto itself. A well-known result of Brodskii and Milman [**1**] asserts that if K is a weakly compact convex subset of E and K has normal structure, then K has the f. p. p. In particular, any compact convex subset of E has the f. p. p. (see [**2**, Lemma 1]).

We consider in this paper bounded closed subsets of a Banach space E with the f. p. p. We prove, among other things, that any bounded closed subsets of E with uniform relative normal structure have the f. p. p. We also show that if E is either the space of bounded complex valued functions on a nonempty set X with the supremum norm (denoted by $l_\infty(X)$), or the algebra of bounded operators on a Hilbert space H (denoted by $\mathfrak{B}(H)$), then E is finite dimensional if and only if the set of states (or normal states) on E has the f. p. p. This is also equivalent to each weak*-compact convex space of E has the f. p. p. Our proofs depend on several well-known properties of discrete amenable groups.

2. Preliminaries and some notations. If E is the Banach space $l_\infty(X)$ or $\mathfrak{B}(H)$ as defined in §1, then E has a unique predual E_* (which is $l_1(X)$ in case $E = l_\infty(X)$, see [**11**] for details). A linear functional ϕ on E is called a *state* if ϕ is positive and $\|\phi\| = 1$; ϕ is *normal* if $\phi \in E_*$.

If G is a group, then G is *amenable* if there exists a state ϕ on $l_\infty(G)$ such that $\phi(l_a f) = \phi(f)$ for each $a \in G$, where $l_a f(X) = f(ax)$, $x \in G$. As is well known, any group containing the free group on two generators is not amenable (see [**8**, p. 236]).

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If E is a Banach space, then the closed ball centre at $x \in E$ and radius $r > 0$ will be denoted by $B[x, r]$. We list below some subsets of E that are known to have the fixed point property:

(1) Any compact convex sets in E.

(2) Any closed ball $B = B[x_0, r]$. In this case x_0 is a common fixed point for any isometric mapping U from B onto B. Indeed, if $U(x_0) = x$, $x \neq x_0$, define $y = x_0 - t(x_0 - x)$, where $t = r/\|x_0 - x\|$, then $y \in B$ and $\|x - y\| = \|x - x_0\| + r > r$. However, if $w \in B$ is such that $U(w) = y$, then $\|x - y\| = \|U(x_0) - U(w)\| = \|x_0 - w\| \leq r$, which is impossible.

(3) Any weakly compact convex set K in E if E is strictly convex. In this case, any isometric mapping from K onto K is necessarily affine [3, Proposition 2]. Apply now the Ryll-Nardzewski fixed point theorem [7, p. 98].

It is not known whether there exist a weakly compact convex subset K of a Banach space and a fixed point free isometry of K onto K (see [4] for a discussion of this open problem).

3. The main results. A closed subset K of a Banach space E is said to have *uniform relative normal structure* if there exists $0 < c < 1$ such that for any nonvoid bounded closed subset M in K, there exist z_M in K such that

(i) $\|x - z_M\| \leq c\delta(M)$ for each $x \in M$,

(ii) if $y \in K$ such that $\|x - y\| \leq c\delta(M)$ for all $x \in M$, then

$$\|z_M - y\| \leq c\delta(M).$$

(Here $\delta(M)$ denotes the diameter of M.) The notion of uniform relative normal structure was introduced recently by P. M. Soardi [11]. He proved, among other things, that if K is a nonempty weak*-closed set with uniform relative normal structure, and T: $K \to K$ is nonexpansive and leaves invariant a weak*-compact subset M in K, then K contains a fixed point for T. Examples of sets with uniform relative normal structure include any closed balls in $L^\infty(X, \mathcal{S}, \mu)$ of a σ-finite measure space $(X, \mathcal{S}, \mu)$ (see [11, p. 28]).

THEOREM 1. *Let E be a Banach space and let K be a bounded closed nonempty subset of E with uniform relative normal structure, then K has the f.p.p.*

PROOF. Let $\mathfrak{U}$ denote the group of isometric mappings from K onto K and let $0 < c < 1$ be a real number satisfying conditions (i) and (ii) for uniform relative normality of K. Put $A_0 = K$, and let $r = \delta(K)$. If $r = 0$, the conclusion is trivial. Otherwise define

$$A_1 = \{k \in A_0;\ A_0 \subseteq B[k, cr]\}.$$

Then A_1 is a closed nonempty subset of K since A_1 is the intersection of A_0 with closed balls and $z_{A_0} \in A_1$ by (i). Also $U(A_1) = A_1$ for each $U \in \mathfrak{U}$. Indeed, if $h \in A_1$ and $x \in A_0$, let $y \in A_0$ such that $U(y) = x$. Then

$$\|U(h) - x\| = \|U(h) - U(y)\| = \|h - y\| \leq cr.$$

Hence $U(H_0) \subseteq H_0$. Equality follows by replacing U with U^{-1}. Moreover $\delta(A_1) < cr$. Hence the set

$$H_1 = \{k \in K; A_1 \subseteq B[k, c^2r]\}$$

is again closed, nonempty and $U(H_1) = H_1$ for each $U \in \mathfrak{U}$. Define

$$A_2 = \{h \in H_1; H_1 \subseteq B[h, c^2r]\}.$$

Then A_2 is closed, nonempty (since $z_{A_1} \in A_2$ by condition (ii)) $\delta(A_2) < c^2r$ and $U(A_2) = A_2$ for each $U \in \mathfrak{U}$. Repeating this process, we have defined a sequence of nonempty closed sets $\{A_n\}_{n=1}^{\infty}$ in K with the following properties:

(i) $U(A_n) = A_n$ for each $U \in \mathfrak{U}$,

(ii) $\delta(A_n) < c^n r$,

(iii) $\|x - y\| < c^n r$ if $x \in A_{n-1}$ and $y \in A_n$.

For each $n = 1, 2, \ldots,$ pick $z_n \in A_n$. Then, as readily checked, $\{z_n\}$ is Cauchy, and $U(z) = z$ for each $U \in \mathfrak{U}$ if z is the limit point of $\{z_n\}$ in K.

THEOREM 2. *Let H be a Hilbert space. Then the following are equivalent*:

(a) *H is finite dimensional.*

(b) *The set of states on $\mathfrak{B}(H)$ has the f.p.p.*

(c) *The set of normal states on $\mathfrak{B}(H)$ has the f.p.p.*

(d) *Each weak*-compact convex subset of $\mathfrak{B}(H)$ has the f.p.p.*

PROOF. If H is finite dimensional, then each of the sets in (b), (c) and (d) is compact and convex, and hence has the f. p. p.

Let $\mathfrak{U}(H)$ denote the group of unitary elements in $\mathfrak{B}(H)$. For each $u \in \mathfrak{U}(H)$, define a weak*-weak* continuous isometric linear map from $\mathfrak{B}(H)$ onto $\mathfrak{B}(H)$ by $\tau_u(x) = u^*xu$, $x \in \mathfrak{B}(H)$. If H is infinite dimensional, write $H = l_2(G)$ where G is a free group, and $|G|$ = the cardinality of a complete orthonormal basis for H. For each $g \in G$, the operator l_g: $l_2(G) \to l_2(G)$ defined by $l_g h(t) = h(gt)$, $t \in G$, is in $\mathfrak{U}(H)$.

Note that each of the sets in (b) and (c) is invariant under each τ_u^*, $u \in \mathfrak{U}(H)$. Hence if (b) or (c) holds, then there exists a state ϕ on $\mathfrak{B}(H)$ such that $\tau_u^*\phi = \phi$ for each $u \in \mathfrak{U}(H)$. For each $f \in l_\infty(G)$, define x_f: $l_2(G) \to l_2(G)$ by $x_f(h) = f \cdot h$ (pointwise multiplication), $h \in l_2(G)$. Then $\|x_f\| < \|f\|$. Define $m \in l_\infty(G)^*$ by

$$m(f) = \phi(x_f), \qquad f \in l_\infty(G).$$

Then, as readily checked, m is a state on $l_\infty(G)$. Also, if $g \in G$, $f \in l_\infty(G)$ then $l_g x_f l_{g^{-1}} = x_k$ where $k = l_g f$. Hence

$$m(f) = \phi(x_f) = \phi(l_g x_f l_{g^{-1}}) = m(l_g f),$$

i.e. G is amenable, which is impossible. Hence H is finite dimensional.

Finally if (d) holds, and H is infinite dimensional, let $VN(G)$ denote the weak*-closure of the linear span of $\{l_g; g \in G\}$ in $\mathfrak{B}(H)$. For each $x \in \mathfrak{B}(H)$, let $K(x)$ be the weak*-closure of the convex hull of $\{l_{g^{-1}} x l_g, g \in G\}$. Then $K(x)$ is weak*-compact and invariant under each τ_u, when $u = l_g$, $g \in G$. Hence there exists $z \in K(x)$ such that $l_{g^{-1}} z l_g = z$ for each $g \in G$. In particular, z is in the commutant of $VN(G)$. By [**10**, Proposition 4.4.21], G must be amenable, which is impossible.

THEOREM 3. *Let X be a nonempty set. Then the following are equivalent*:

(a) *X is finite.*

(b) *The set of states on $l_\infty(X)$ has the f.p.p.*

(c) *The set of normal states on $l_\infty(X)$ has the f.p.p.*

(d) *Each weak*-compact convex subset of $l_\infty(X)$ has the f.p.p.*

PROOF. If X is finite, then $l_\infty(X)$ and $l_\infty(X)^*$ are finite dimensional, and hence (b), (c) and (d) hold.

If X is infinite, then we may regard X as a free group G on $|X|$-generators. Also, the set of states (or normal states) on $l_\infty(G)$ is invariant under each of the isometrics l_g^*, $g \in G$. Hence if (b) or (c) holds, then there exists a state ϕ on $l_\infty(G)$ such that $l_g^*\phi = \phi$ for all $g \in G$, which is impossible. Finally, if (d) holds, then for each $f \in l_\infty(G)$, define $(r_g f)(x) = f(xg)$ for each $x, a \in G$. Let $K(f)$ be the weak*-closed convex hull of $\{r_g f;\ g \in G\}$. Then $K(f)$ is also weak*-compact. Hence by (d), there exists $h \in K(f)$, such that $r_g h = h$ for all $g \in G$. Necessarily h is a constant. By a result of Mitchell [**9**, Corollary 2], G is amenable.

THEOREM 4. *Let X be a nonempty set. Then the following are equivalent*:

(a) *X has one element.*

(b) *The set of nonzero multiplicative linear functionals on $l_\infty(X)$ has the f.p.p.*

(c) *Each weak*-compact subset of $l_\infty(X)$ has the f.p.p.*

PROOF. That (a) implies (b) and (c) is trivial. To prove (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a), regard X as a group. Then the set of nonzero multiplicative linear functionals on $l_\infty(G)$ is invariant under each of the isometric mappings l_g^*, $g \in G$. Now if (b) holds, then there exists a nonzero multiplicative linear functional ϕ on $l_\infty(G)$ such that $l_g^*\phi = \phi$ for all $g \in G$, which implies G is trivial by [**5**, Corollary 3]. If (c) holds, then an argument similar to that of Theorem 3 shows that for each f in $l_\infty(G)$, the weak*-closure of $\{r_g f;\ g \in G\}$ contains a constant, which again implies that G is trivial by [**6**, Theorem 1] and [**5**, Corollary 3].

REMARK. Let E be a dual Banach space, then each closed ball in E is weak*-compact convex and has the f. p. p. Also, if E is finite dimensional, then each weak*-compact convex subset of E has the f. p. p. However, the converse is false unless $E = l_\infty(X)$, or $E = \mathfrak{B}(H)$ by Theorems 2 and 3. (For example, take E to be an infinite dimensional Hilbert space.)

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA T6G 2G1, CANADA