VALENCE SEQUENCES

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ABSTRACT. Let $V_0, V_1, \dots, V_k, \dots$ be an infinite sequence of positive integers (where we include ∞ as a possible value for V_k). The sequence is called a *valence sequence* if there is an $f(z)$ regular in the unit disk such that for each $k > 0$ the k th derivative $f^{(k)}(z)$ has valence V_k in E . Most of the results obtained here about valence sequences are obvious, but we prove three theorems about valence sequences that are not trivial.

1. The definition of a valence sequence. Let $V_0, V_1, V_2, \dots, V_k, \dots$ be an infinite sequence of positive integers. We say that $\{V_k\}_0^\infty$ is a *valence sequence* if there is an $f(z)$ regular in $E: |z| < 1$ such that $f^{(k)}(z)$ has valence V_k in E for each $k > 0$. Since any constant function is infinite valent we must include ∞ as a possible value for V_k . Each function analytic in E gives rise to a valence sequence. For example if $f(z) = z + z^3/3$ then the associated valence sequence is $1, 2, 1, \infty, \infty, \dots$. The main problem is to determine all valence sequences, but this seems to be very difficult. In this paper we look at a few properties of valence sequences and find a few sufficient conditions for a sequence to be a valence sequence.

2. Some simple examples. As Shah and Trimble [6] have observed, if $0 < c < \pi$ and $f(z) \equiv e^{cz}$ then $f(z)$ and all of its derivatives are univalent in E . Thus the sequence $\{1\}_0^\infty = 1, 1, 1, \dots$ is a valence sequence. More generally any constant sequence $\{p\}$ where p is a positive integer is a valence sequence. We merely select c to be a suitable real constant in $f(z) = e^{cz}$. The function $f(z) \equiv c$ gives rise to the sequence $\infty, \infty, \dots, \infty, \dots$, but we regard this as a trivial example of no real interest. In §3 we will give a nonconstant function $f(z)$ for which $f^{(k)}(z)$ has infinite valence for each integer $k > 0$. Thus every constant sequence of positive integers is a valence sequence. The function $f(z) = 1/(1+z)$ shows that $1, 1, 2, 2, 3, 3, \dots$ is a valence sequence. Here we can set $V_k = 1 + [k/2]$. A study of the mapping properties of $\sin z$ and $\cos z$ show that if $0 < c < \pi/2$ then $\sin cz$ has the sequence $1, 2, 1, 2, \dots$ and if $\pi < c < 3\pi/2$, then $\sin cz$ has the sequence $3, 4, 3, 4, \dots$. Generally if $(p-1)\pi < c < (2p-1)\pi/2$ then $V_k = 2p-1 + [1 + (-1)^k]/2$ for $f(z) = \sin cz$.

We remark that if $\{V_n\}_0^\infty$ is a valence sequence, then (by starting with $F(z) = f^{(k)}(z)$) the sequence obtained by dropping the first k terms, $\{V_{k+n}\}_{n=0}^\infty$ is also a valence sequence. Thus $f(z) = 1/(1+z)$ and $\sin cz$ generate new valence sequences.

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3. More valence sequences. Here we give two theorems on valence sequences that are not as obvious as those in §2. If $f(z) = z^p$ then the sequence starts with $p, p-1, \dots, 2, 1$. Our object is to obtain an infinite sequence made up of sections of this type.

We say that a sequence $\{V_n\}_0^\infty$ is *piecewise slowly decreasing to one*, if each V_n is a positive integer and if for each n either $V_{n+1} = V_n - 1$ or $V_n = 1$ and $V_{n+1} > 1$.

THEOREM 1. *Each sequence that is piecewise slowly decreasing to one is a valence sequence.*

PROOF. In general we set

$$f(z) = \sum_{n=1}^{\infty} a_n z^n. \quad (1)$$

A. Hurwitz [3] was the first to remark that if $|a_1| > \sum_{n=2}^{\infty} n|a_n|$, then $f(z)$ is univalent in E . S. Ozaki [5] generalized this condition as follows. If

$$f(z) = a_p z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad a_p \neq 0, \quad (2)$$

and if

$$p|a_p| > \sum_{n=p+1}^{\infty} n|a_n| \quad (3)$$

then $f(z)$ is p -valent in E . We will give Ozaki's proof of this theorem in §6.

Now suppose that for some N our sequence contains the integers

$$V_{N-1} = 1, \quad V_N = p, \quad V_{N+1} = p-1, \dots, V_{N+p-1} = 1, \quad (4)$$

where $N > 1$ and $p > 1$. We must produce a function $F(z)$ for which $F^{(k)}(z)$ has valence V_k for $k = N-1, N, \dots, N+p-1$. We begin with the series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} z^n \quad (5)$$

and we delete those terms for which $n = N+1, N+2, \dots, N+p-1$ to obtain a new function $F(z)$. Then

$$F^{(N-1)}(z) - \frac{1}{(N-1)!} = \frac{z}{N!} + \sum_{n=p+1}^{\infty} \frac{z^n}{(n+N-1)!n!}. \quad (6)$$

We apply the Hurwitz condition to $F^{(N-1)}(z)$. Since

$$\sum_{n=p+1}^{\infty} \frac{nN!}{(n+N-1)!n!} < \sum_{n=2}^{\infty} \frac{1}{2(n-1)!} = \frac{e-1}{2} < 1 \quad (7)$$

for each $N > 1$ and $p > 1$, we see that $F^{(N-1)}(z)$ is univalent in E .

Now consider $F^{(N+k)}(z)$ for $k = 0, 1, 2, \dots, p-1$. We are to prove that $F^{(N+k)}(z)$ is $(p-k)$ -valent in E . Now

$$F^{(N+k)}(z) - F^{(N+k)}(0) = \frac{z^{p-k}}{(N+p)!(p-k)!} + \sum_{n=p-k+1}^{\infty} \frac{z^n}{(n+N+k)!n!} \quad (8)$$

and so by the Ozaki condition we are to prove that

$$\sum_{n=p-k+1}^{\infty} \frac{n}{(n+N+k)!n!} < \frac{p-k}{(N+p)!(p-k)!}. \quad (9)$$

If we set $p-k=d=1, 2, 3, \dots, p$ and make a few standard changes in the indices, inequality (9) is equivalent to

$$\sum_{n=1}^{\infty} \frac{(N+p)!}{(n+N+p)!} \cdot \frac{(d-1)!}{(n+d-1)!} < 1. \quad (10)$$

But for $N > 1, p > 1, d > 1$, the left side is less than $(e-1)/3$.

Suppose now that the sequence $\{V_k\}_0^{\infty}$ is composed of infinitely many pieces of the form (4). We start with the series (5) and delete a suitable set of terms for each piece. These deletions merely strengthen the inequalities (7) and (9) by decreasing the left hand side. Hence $F^{(k)}(z)$ has valence V_k for $k=0, 1, 2, \dots$. $\square$

THEOREM 2. *There is a function $F(z)$, not a constant, such that $F^{(k)}$ has valence ∞ for each positive integer k .*

PROOF. Let

$$\begin{aligned} F(z) &= (A+Bi)(1+z)^{a+bi} + (A-Bi)(1+z)^{a-bi} \\ &\equiv (A+Bi)e^{(a+bi)\ln(1+z)} + (A-Bi)e^{(a-bi)\ln(1+z)} \end{aligned} \quad (11)$$

where $\ln(1+z)$ is the principal branch in E . We assume that A, B, a , and b are nonzero real numbers with $b > 0$. For convenience set $\theta_0 = \text{Arg}(A+Bi)$ and for each integer n define z_n and c_n by

$$\ln(1+z_n) = (\pi/2 - 2n\pi - \theta_0)/b \equiv c_n. \quad (12)$$

Clearly there are infinitely many integers n for which $c_n < 0$. For each such n we have $z_n = e^{c_n} - 1$ and hence $-1 < z_n < 0$. Since

$$\begin{aligned} (A+Bi)e^{(a+bi)\ln(1+z_n)} &= \sqrt{A^2+B^2} e^{i\theta_0} e^{ac_n} e^{i(\pi/2-\theta_0)} \\ &= \sqrt{A^2+B^2} e^{ac_n} i, \end{aligned}$$

and

$$\begin{aligned} (A-Bi)e^{(a-bi)\ln(1+z_n)} &= \sqrt{A^2+B^2} e^{-i\theta_0} e^{ac_n} e^{-i(\pi/2-\theta_0)} \\ &= -\sqrt{A^2+B^2} e^{ac_n} i, \end{aligned}$$

we see that $F(z_n) = 0$ for infinitely many z_n in E .

Finally if we start with a function of the form (11), then the derivative has the same form and is also infinite-valent in E . Thus this $F(z)$ has the valence sequence $\{\infty\}_0^{\infty}$. $\square$

We observe that functions similar to the one defined in (11) have been used previously in the literature [1], [2], and [4].

4. Adjacent pairs in a valence sequence. In this section we show that any pair of integers (including ∞) may occur as adjacent elements in a valence sequence. Without loss of generality it is sufficient to prove the result for V_0 and V_1 .

THEOREM 3. *For any pair of positive integers p and q , there is a function $F(z)$ that is p -valent in E and for which $F'(z)$ is q -valent in E . Here either p or q (or both) may be infinity.*

For brevity we let (p, q) denote the infinite sequence $(p, q, \dots)$ where the remaining elements are not specified, and we call (p, q) a finite sequence.

PROOF. Set

$$F(z) = (1 + z)^{a+bi} \equiv e^{(a+bi)\ln(1+z)} \quad (13)$$

where a and b are both positive. We also impose the condition $0 < a < 1$. A consideration of the mapping properties of e^z and $\ln(1 + z)$ show that if

$$p - 1 < s \equiv (a^2 + b^2)/2a < p, \quad (14)$$

where p is a positive integer, then $F(z)$ is p -valent in E . For the derivative, $F'(z) = (a + bi)(1 + z)^{a-1+bi}$, we observe that $a - 1 < 0$ and hence the condition for $F'(z)$ to be q -valent in E takes the form

$$q - 1 < t \equiv ((1 - a)^2 + b^2)/2(1 - a) < q. \quad (15)$$

If we solve the pair of equations (14) and (15) for a and b we find that

$$a = (2t - 1)/2(s + t - 1), \quad (16)$$

and

$$b^2 = (2t - 1)[4s(s + t - 1) - 2t + 1]/4(s + t - 1)^2. \quad (17)$$

If we require that $s > 1/2$ and $t > 1/2$, then $b^2 > 0$ and $0 < a < 1$. Now given p and q finite positive integers, we can select $s > 1/2$ and $t > 1/2$ subject to the restrictions (13) and (14). We determine a and $b > 0$ from equations (16) and (17). With these selections $F(z) = (1 + z)^{a+bi}$ is p -valent in E and $F'(z)$ is q -valent in E .

The sequence (∞, ∞) is a special case of Theorem 2.

To generate the sequence (p, ∞) , set $a = 1$ and determine b so that $(1 + b^2)/2 = p$ (see the inequality (14)).

To generate the sequence (∞, q) , set $a = 0$ and determine b so that $(1 + b^2)/2 = q$. $\square$

5. Open problems. We can also consider valence sequences under the operation of integration. If we are looking at finite sequences, we obtain nothing new, because every finite valence sequence under integration is the reverse of a finite valence sequence under differentiation. However the problem of finding infinite valence sequences under integration may be quite different from the problem for differentiation.

Suppose that $\{U_k\}_0^\infty$ and $\{V_k\}_0^\infty$ are valence sequences. Is it true that the sum $\{U_k + V_k\}_0^\infty$ is also a valence sequence? As a special case let $\{U_k\}_0^\infty$ be a piecewise slowly decreasing to one sequence and let $\{p\}_0^\infty$ be a constant sequence. Is this sum a valence sequence? If we multiply a valence sequence $\{V_k\}_0^\infty$ by a natural number p , is the result $\{pV_k\}_0^\infty$ a valence sequence? Is there a valence sequence in which infinitely many of the terms are finite and infinitely many of the terms are infinite? Is there a periodic valence sequence for which $|V_{k+1} - V_k| = 2$ for each $k > 0$? Is

the sequence $1, 2, 3, \dots = \{n+1\}_0^\infty$ a valence sequence? Can we specify any two elements of a valence sequence in advance; in other words, given positive integers p, q, N , and k , is there a function $F(z)$ for which $V_N = p$ and $V_{N+k} = q$?

6. The Ozaki condition for p -valence. It is well known that if $f(z) = a_p z^p + \sum_{n=p+1}^\infty a_n z^n$ ($a_p \neq 0$) satisfies the inequality $\operatorname{Re}[zf'(z)/f(z)] > 0$ in E , then $f(z)$ is generalized starlike and hence p -valent in E . Now suppose that $p|a_p| > \sum_{n=p+1}^\infty n|a_n|$. Then (where all sums run from $p+1$ to ∞)

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \frac{pa_p z^p + \sum na_n z^n}{a_p z^p + \sum a_n z^n} = \frac{pa_p + \sum na_n z^{n-p}}{a_p + \sum a_n z^{n-p}} \\ &= p + \frac{\sum(n-p)a_n z^{n-p}}{a_p + \sum a_n z^{n-p}}. \end{aligned}$$

Hence in E

$$\begin{aligned} \operatorname{Re} \frac{zf'(z)}{f(z)} &> p - \operatorname{Re} \frac{\sum(n-p)a_n z^{n-p}}{a_p + \sum a_n z^{n-p}} > p - \frac{\sum(n-p)|a_n| r^{n-p}}{|a_p| - \sum |a_n| r^{n-p}} \\ &> \frac{p|a_p| - \sum n|a_n| r^{n-p}}{|a_p| - \sum |a_n| r^{n-p}} > 0. \end{aligned}$$

Consequently $f(z)$ is p -valent in E .

REFERENCES

1. D. M. Campbell and V. Singh, *Valence properties of the solution of a differential equation*, Pacific J. Math. (to appear).
2. A. W. Goodman, *The valence of certain means*, J. Analyse Math. **22** (1969), 355–361.
3. A. Hurwitz, *Über die Entwicklung der allgemein Theorie der analytischen Funktionen in neuerer Zeit*, Mathematische Werke, Vol. 1, Birkhauser, Basel, 1932, pp. 461–480 (see footnote 16, p. 478).
4. J. Krzyz and Z. Lewandowski, *On the integral of univalent functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 447–448.
5. S. Ozaki, *Some remarks on the univalence and multivalence of functions*, Sci. Rep. Tokyo Bunrika Daigaku Sect. A **2** (1934), 41–55.
6. S. M. Shah and S. Y. Trimble, *Univalent functions with univalent derivatives*, Bull. Amer. Math. Soc. **75** (1969), 153–157.

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