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A HOLONOMY PROOF OF THE POSITIVE CURVATURE OPERATOR THEOREM[1]

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ABSTRACT. Extending work of Bochner-Yano and M. Berger, D. Meyer proved that if the curvature operator of a compact, oriented, Riemannian manifold M has positive eigenvalues, then M is a rational homology sphere. Here a proof is given using Chern's holonomy formula for the Laplacian on M; for completeness, a quick proof of Chern's formula is included.

Chern [3] showed that the Hodge theory of compact Kähler manifolds is a special case of a general phenomenon; by reinterpreting Weizenböck's formula for the Laplacian of a Riemannian manifold in terms of the holonomy group of the manifold, he was able to simplify many of the classical cohomology calculations on Kähler manifolds, while simultaneously exhibiting other geometric structures for which similar theorems hold.

In this note Chern's formula is used to simplify the proof of the following result.

THEOREM. *If M^n is an oriented, compact Riemannian manifold with positive definite curvature operator, then M is a rational homology sphere.*

Bochner and Yano [2], [7] proved that if the eigenvalues of the curvature operator of M lie in the interval $[\frac{1}{2}, 1]$, then by Weizenböck's formula,

$$\int_M \langle \Delta\mu, \mu \rangle \geqslant \frac{r(n-2r+1)}{2} \int_M \|\mu\|^2$$

for an r-form μ on M. Thus $H^r(M; \mathbf{R})$ is zero for $r = 1, 2, \ldots, n-1$.

Berger [1] observed that for a form μ of degree 2, the integral over M of $\langle \Delta\mu, \mu \rangle$ could be estimated from below in terms of the minimum λ of the eigenvalues of the curvature operator; this proved that $H^2(M; \mathbf{R})$ is zero if the curvature operator of M is positive definite. He then asked whether this hypothesis implies the vanishing of $H^r(M; \mathbf{R})$ for $0 < r < n$.

This question was answered affirmatively by Meyer [6]. Subsequently, Gallot and Meyer [4] obtained the sharp estimate

$$\int_M \langle \Delta\mu, \mu \rangle \geqslant \lambda r(n-r) \int_M \|\mu\|^2 \tag{1}$$

for all r-forms on M, which makes the proof of the theorem completely obvious.

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In this note, inequality (1) will follow from Chern's holonomy formula for the Laplacian by comparison of $\langle \Delta\mu, \mu \rangle$ with the Laplacian on the sphere of curvature λ. For completeness, a quick derivation of Chern's formula is included; cf. also Weil [8].

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1. Chern's formula for the Laplacian. Fix an r-form μ on M. Given a point p in M, vectors $v_1, \ldots, v_r$ in M_p, and an orthonormal basis $\{e_j\}$ for M_p, define

$$\mathcal{R}_\mu(v_1, \ldots, v_r) := \sum_{i=1}^{n} \sum_{j=1}^{r} (R(e_i, v_j)\mu)(v_1, \ldots, v_{j-1}, e_i, v_{j+1}, \ldots, v_r).$$

Weizenböck's formula states that

$$\langle \Delta\mu, \mu \rangle = \tfrac{1}{2}\Delta\|\mu\|^2 + \|\nabla\mu\|^2 + \langle \mathcal{R}_\mu, \mu \rangle. \tag{2}$$

If u and v are tangent vectors on M at p, then it is well known that

$$R(u, v)\mu = -\mu \circ R(u, v) = -{}^tR(u, v)(\mu),$$

where we use the natural extension of $R(u, v) \in \operatorname{End}(M_p)$ to a derivation of $\Lambda_r M_p$, and then take the transpose to get an element of $\operatorname{End}(\Lambda_r M_p^*)$.

Now suppose that the holonomy group of M is the Lie subgroup G of $SO(n)$. The holonomy algebra $\mathfrak{g}(M_p) \subset o(M_p)$ can be embedded into $\Lambda_2 M_p$ by

$$\langle X, u \wedge v \rangle_{\Lambda_2 M_p} := \langle X(v), u \rangle_{M_p}, \qquad X \in \mathfrak{g}(M_p),\ u, v \in M_p.$$

Let $\{X_\alpha\}$ be an orthonormal basis for $\mathfrak{g}(M_p)$; because of the equality $R(u, v) = \Sigma_\alpha \langle R(u, v), X_\alpha \rangle X_\alpha = \Sigma_\alpha \langle R(X_\alpha)v, u \rangle X_\alpha$,

$$\begin{aligned}\mathcal{R}_\mu(v_1, \ldots, v_r) &= -\sum_{i=1}^{n} \sum_{j=1}^{r} \sum_{\alpha} \langle R(X_\alpha)v_j, e_i \rangle {}^tX_\alpha(\mu)(v_1, \ldots, v_{j-1}, e_i, v_{j+1}, \ldots, v_r) \\ &= -\sum_{\alpha} ({}^tR(X_\alpha){}^tX_\alpha\mu)(v_1, \ldots, v_r).\end{aligned}$$

Thus

$$\langle \Delta\mu, \mu \rangle(p) = \left(\tfrac{1}{2}\Delta\|\mu\|^2 + \|\nabla\mu\|^2\right)(p) + \sum_{\alpha} \langle {}^tX_\alpha\mu, {}^tR(X_\alpha)\mu \rangle$$

since $R(X_\alpha)$ is skew-symmetric; this is Chern's formula.

2. Proof of the positive curvature operator theorem. Assume the notation of §1. Let all the eigenvalues of R be greater than or equal to $\lambda > 0$. Fix $p \in M$ and choose an orthonormal basis $\{X_\alpha\}$ for $o(M_p) \cong \Lambda_2 M_p$. Since $\langle R(X_\alpha), X_\alpha \rangle \geqslant \lambda > 0$ for all α,

$$\sum_{\alpha} \langle {}^tR(X_\alpha)\mu, {}^tX_\alpha\mu \rangle \geqslant \lambda \sum_{\alpha} \|{}^tX_\alpha\mu\|^2. \tag{3}$$

It also follows that the holonomy algebra $\mathfrak{g}(M_p)$ equals $o(M_p)$, which allows us to calculate the right-hand side of (3) on any appropriate space. Isometrically identify M_p with the tangent space S^n_q at some point q in the n-sphere S^n; μ_p corresponds to the value η_q of some r-form η on S^n. By [7, §4.1], $\Delta\eta = -\text{div}\,\nabla\eta + r(n-r)\eta$, so

$$\|{}^tX_\alpha\mu_p\|^2 = \langle\Delta\eta + \text{div}\,\nabla\eta, \eta\rangle(q) = r(n-r)\|\eta_q\|^2 = r(n-r)\|\mu_p\|^2.$$

Thus,

$$\langle\Delta\mu, \mu\rangle \geqslant \tfrac{1}{2}\Delta\|\mu\|^2 + \|\nabla\mu\|^2 + \lambda r(n-r)\|\mu\|^2.$$

Integration over M now yields inequality (1).

References

1. M. Berger, *Sur les variétés à opérateur de courbure positif*, C. R. Acad. Sci. Paris Sér A-B **253** (1961), 2832–2834.

2. S. Bochner and K. Yano, *Tensor-fields in non-symmetric connections*, Ann. of Math. (2) **56** (1952), 504–519.

3. S. S. Chern, *On a generalization of Kähler geometry*, Algebraic Geometry and Topology, Sympos. in honor of S. Lefschetz, Princeton Univ. Press, Princeton, N. J., 1959, pp. 103–121.

4. S. Gallot and D. Meyer, *Opérateur de courbure et Laplacien des formes différentielles d'une variété riemannienne*, J. Math. Pures Appl. **54** (1975), 259–284.

5. H. Maillot, *Sur les variétés riemanniennes à opérateur de courbure pur*, C. R. Acad. Sci. Paris Sér. A-B **278** (1974), 1127–1130.

6. D. Meyer, *Sur les variétés riemanniennes à opérateur de courbure positif*, C. R. Acad. Sci. Paris Sér. A-B **272** (1971), 482–485.

7. K. Yano and S. Bochner, *Curvature and Betti numbers*, Ann. of Math. Studies, no. 32, Princeton Univ. Press, Princeton, N. J., 1953.

8. A. Weil, *Un théorème fondamental de Chern en géométrie riemannienne*, Séminaire Bourbaki, 14e année, 1961/62, no. 239.

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