CUTTING FAMILIES OF CONVEX SETS

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ABSTRACT. A family of convex sets in the plane admits a common transversal if there is a straight line which intersects (cuts) each member of the family.

It is shown that there is a positive integer k such that for any compact convex set C in the plane and for any finite family $\mathcal{C}$ of pairwise disjoint translates of C : If each 3-membered subfamily of $\mathcal{C}$ admits a common transversal then there is a subfamily $\mathcal{B}$ of $\mathcal{C}$ such that $\mathcal{B}$ admits a common transversal and $|\mathcal{C} \setminus \mathcal{B}| < k$.

1. Introduction. A family $\mathcal{F}$ of subsets of the Euclidean plane is said to have *property T* , or to admit a *common transversal*, if there is a straight line which intersects every member of $\mathcal{F}$. The family has *property $T(m)$* , m a positive integer, if every m -membered subfamily of $\mathcal{F}$ has property T . Unless otherwise mentioned, we will assume throughout that $\mathcal{F}$ is a finite family of compact convex sets.

Many of the known results about common transversals (cf. [1, Chapter 5] and [8] for some of these facts) are concerned with the problem of finding conditions on $\mathcal{F}$ such that $T(m)$ implies T .

For obvious reasons the smallest value of m for which this problem can be solved is $m = 3$, and there are nontrivial families $\mathcal{F}$ for which $T(3)$ implies T ; eg., if $\mathcal{F}$ consists of parallel straight line segments [11], [12], [13], or if $\mathcal{F}$ consists of infinitely many pairwise disjoint congruent compact sets with nonempty interiors [6]. On the other hand, there are also families for which $T(3)$ does not imply T . In particular, if $\mathcal{F}$ consists of N pairwise disjoint translates of a parallelogram, then $T(5)$ implies T and even $T(4)$ does not [4]. In this case, however, $T(4)$ does imply that there is a straight line intersecting $N - 2$ members of $\mathcal{F}$ [9], and one is naturally led to ask if a similar conclusion can be drawn if it is assumed only that $\mathcal{F}$ has $T(3)$. In this note we show that such a conclusion is valid even without the assumption that the members of $\mathcal{F}$ are parallelograms.

We will say that a family $\mathcal{F}$ of subsets of the plane has *property $T - k$* , k a nonnegative integer, if there is a straight line intersecting all but at most k members of $\mathcal{F}$. Using this terminology, our main result is:

1. THEOREM. *There exists a positive integer k such that for any family $\mathcal{F}$ of pairwise disjoint translates of a compact convex set C , $T(3)$ implies $T - k$.*

Theorem 1 will follow from the more general

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2. **THEOREM.** Let $\mathcal{F}$ be any family of compact, convex subsets in the plane, and suppose that each member of $\mathcal{F}$ has diameter no greater than $r > 0$. If $\mathcal{F}$ has $T(3)$, there are 3 discs of radius $3r$ such that there is a common transversal for all members of $\mathcal{F}$ which do not intersect any of the discs.

A simple corollary of Theorem 2 which will be of use is

3. **COROLLARY.** Given $r > 0$, $\beta > 0$ and $n > 0$, there is a positive integer $l = l(r, \beta, n)$ such that $T(3)$ implies $T - l$ for any family $\mathcal{F}$ of compact convex sets satisfying the following conditions:

- (i) each member of $\mathcal{F}$ has diameter no greater than r ;
- (ii) for every n -membered subfamily $\{A_1, \dots, A_n\}$ of $\mathcal{F}$, the area of $\bigcup_{i=1}^n A_i$ is no smaller than $\beta \cdot n$.

The integer k in Theorem 1 is universal for families of pairwise disjoint translates of a compact convex set. It is interesting to note that, for such families, it is unknown whether there is also a universal m such that $T(m)$ implies T . (It is a long standing conjecture of Grünbaum [1], [4] that $m = 5$ is universal.) In this regard, we also mention that for families $\mathcal{F}$ of arbitrary compact convex sets, there is no universal m [8], not even when $\mathcal{F}$ consists of pairwise disjoint congruent rectangles [10].

2. **Proofs.** We require the following somewhat technical lemma.

4. **LEMMA.** Let $\{A_1, B, A_2\}$ be a family of three rectangles satisfying the following conditions:

- (i) the edges are of length no greater than r and parallel to the coordinate axes;
- (ii) the distance between each two rectangles of the family is greater than r ;
- (iii) the horizontal coordinate axis h intersects the three rectangles in the order A_1, B, A_2 , and separates $A_1 \cup A_2$ from B . (See Figure 1.)

FIGURE 1

If D is any rectangle satisfying (i) and if the family $\{A_1, B, A_2, D\}$ satisfies (ii) and also has property $T(3)$, then D intersects the horizontal axis. (By the distance between two subsets X and Y mean $\inf\{\text{dist}(x, y) : x \in X, y \in Y\}$.)

PROOF. We will show that D can be neither strictly above h nor strictly below h . Referring to Figure 1, we note that if D lies above h , then it must intersect the

vertical strip defined by the extension of the vertical edges of B . (If, for example it were to the left, then $\{A_2, B, D\}$ would fail T .) Then, in order that $\{A_1, A_2, D\}$ have property T , D cannot lie above the straight line through the uppermost horizontal edge of $A_1 \cup A_2$. But then $\text{dist}(D, B) < r$, a contradiction. On the other hand, again referring to Figure 1, if D lies strictly below h then it must intersect one of two vertical half strips, the one below A_1 , the other below A_2 . (If D lies between these strips, then $\{A_1, A_2, D\}$ fails T ; if D lies to the right of the strip below A_2 , then $\{B, A_2, D\}$ fails T .) Assuming then that D intersects the strip below A_1 , we note that D cannot lie strictly below the straight line containing the lower horizontal edge of B , for this would imply that $\{A_1, B, D\}$ fails T . But then, the distance between A_1 and D is no greater than r , a contradiction. This completes the lemma.

PROOF OF THEOREM 2. Letting $\mathcal{F} = \{C(\gamma) : \gamma \in \Gamma\}$ be the family, choose $x(\gamma) \in C(\gamma)$ for each $\gamma \in \Gamma$, and contract each $C(\gamma)$ about the point $x(\gamma)$ by a factor of $\lambda \in [0, 1]$. Let λ_0 be the minimum value of λ such that, for the contracted family $\mathcal{G} = \{C'(\gamma) : \gamma \in \Gamma\}$, there is a common transversal for every three members $C'(\gamma_1), C'(\gamma_2), C'(\gamma_3)$ whenever $\text{dist}(C(\gamma_i), C(\gamma_j)) > (1 + \sqrt{2})r$, $1 \leq i \neq j \leq 3$. (If some three members of $\mathcal{G}$ are mutually separated by distances greater than $(1 + \sqrt{2})r$, then the existence of λ_0 can be established by standard compactness arguments. On the other hand if no three members of $\mathcal{G}$ are mutually separated by distances greater than $(1 + \sqrt{2})r$, then the theorem is trivially satisfied for the following reasons: either there is some pair $\{C'(\gamma_1), C'(\gamma_2)\}$ whose members are separated by a distance greater than $(1 + \sqrt{2})r$, or there is no such pair. In the latter case, let $\{C'(\gamma_1), C'(\gamma_2)\}$ be any pair chosen from $\mathcal{G}$. Each $C'(\gamma_i)$ has diameter no greater than r , so by Jung's theorem (cf. [1, p. 112]) is contained in a disc of radius at most $r/\sqrt{3}$. Every other member of $\mathcal{G}$ must be with a distance $(1 + \sqrt{2})r$ from at least one of these two discs, and therefore every member of $\mathcal{G}$ intersects at least one of two discs of radius $(1 + \sqrt{2} + 1/\sqrt{3})r < 3r$.)

By the minimality of λ_0 , there are three members of $\mathcal{G}$, say $C'(\gamma_1), C'(\gamma_2), C'(\gamma_3)$ and a straight line h intersecting the three sets in the order listed with h separating $C'(\gamma_1) \cup C'(\gamma_3)$ from $C'(\gamma_2)$. (See Figure 2.) Let R_i be the smallest rectangle, with one pair of edges parallel to h , which contains $C'(\gamma_i)$, $i = 1, 2, 3$, and set $R_1 = A_1$, $R_2 = B$, $R_3 = A_2$.

FIGURE 2

Assuming, without loss of generality, that h is the horizontal axis and that B lies below h , we claim that A_1, B , and A_2 satisfy all of the hypotheses of Lemma 4.

Since the diameter of $C'(\gamma_i)$ is no greater than r , it is clear that (i) is satisfied, and (ii) is a consequence of the fact that $\text{dist}(C'(\gamma_i), C'(\gamma_j)) > (1 + \sqrt{2})r$.

Now suppose that $C'(\gamma)$ is any member of $\mathcal{G}$ such that $\text{dist}(C'(\gamma), C'(\gamma_i)) > (1 + \sqrt{2})r$ for $i = 1, 2, 3$. (If there is no such $C'(\gamma)$, then every member of $\mathcal{G}$ intersects at least one of three fixed discs of radii $3r$, each of these discs containing one of $C'(\gamma_i)$, $i = 1, 2, 3$.) Let D be the smallest rectangle with edges parallel to h which contains $C'(\gamma)$. Then, as above, $\text{dist}(D, A_1) > r$, $\text{dist}(D, A_2) > r$, $\text{dist}(D, B) > r$, and the family $\{A_1, B, A_2, D\}$ has $T(3)$ since $\{C'(\gamma_1), C'(\gamma_2), C'(\gamma_3), C'(\gamma)\}$ does. Then, D , and therefore $C'(\gamma)$, intersects h , completing the proof of Theorem 2.

The corollary now follows from Theorem 2: condition (ii) of the corollary implies that, with the possible exception of $(n - 1)$ members of $\mathcal{F}$, every member has area at least β . Thus three fixed discs of radius $4r$ can contain at most $48\pi r^2 \beta^{-1} + (n - 1)$ members of $\mathcal{F}$. From Theorem 2, we have $l(r, \beta, n) = k \leq 48\pi r^2 \beta^{-1} + (n - 1)$.

PROOF OF THEOREM 1. For families of parallel segments, $T(3)$ implies T , so we may assume that C has nonempty interior. In other words, each member of $\mathcal{F}$ has diameter r and every 1-membered subfamily of $\mathcal{F}$ has area β , and Theorem 1 will be an immediate consequence of the corollary once it is shown that we can always arrange things so that $r = \sqrt{2}$ and $\beta = \frac{1}{2}$. To do this, first enclose C in a parallelogram A in such a manner that the area of C is at least one-half the area of A (cf. Lemma 2 of [2]). Now let $\mathcal{L}$ be a linear transformation mapping A into a square S whose edges have length 1. Then $\mathcal{L}(C)$ has area at least $\frac{1}{2}$ and diameter at most $\sqrt{2}$. Thus, $\mathcal{L}(\mathcal{F})$ (and hence $\mathcal{F}$) has property $T - k$ where $k = l(\sqrt{2}, \frac{1}{2}, 1)$, completing the proof of Theorem 1.

REMARKS. (i) Using $l(r, \beta, n) = l(\sqrt{2}, \frac{1}{2}, 1)$ provides an upper bound of 192π for the value of k in Theorem 1, and it seems clear that there is room for improvement. This seems much too large, and we in fact conjecture that the value $k = 2$ is universal for families of disjoint translates of a compact convex set. However, even when the members of $\mathcal{F}$ are squares, or circles, it is not known whether $k = 2$.

(ii) It is readily seen that disjointedness is necessary in Theorem 1.

(iii) Using a construction similar to that in [10], an example using congruent straight line segments will show that the value of k in Theorem 1 does not hold if we allow rotations as well as translations. In other words, there is no universal k for families consisting of pairwise disjoint congruent convex sets.

REFERENCES

1. L. Danzer, B. Grünbaum and V. Klee, *Helly's theorem and its relatives*, Convexity, Proc. Sympos. Pure Math., vol. 7, Amer. Math. Soc., Providence, R. I., 1963, pp. 101–180.
2. Th. Estermann, *Über den Vektorenbereich eines Konvexen Körpers*, Math. Z. **28** (1928), 471–475.
3. B. Grünbaum, *On a theorem of L. A. Santaló*, Pacific J. Math. **5** (1955), 351–359.
4. ———, *On common transversals*, Arch. Math. **9** (1958), 465–469.
5. ———, *Common transversals for families of sets*, J. London Math. Soc. **35** (1960), 408–416.
6. H. Hadwiger, *Über einen Satz Hellyscher Art*, Arch. Math. **7** (1956), 377–379.
7. ———, *Über Eibereiche mit gemeinsamer Treffgeraden*, Portugal. Math. **16** (1957), 23–29.

8. H. Hadwiger, H. Debrunner and V. Klee, *Combinatorial geometry in the plane*, Holt, Rinehart and Winston, New York, 1964.
9. M. Katchalski and T. Lewis, *Cutting rectangles in the plane* (to appear).
10. T. Lewis, *Two counterexamples concerning transversals for convex subsets of the plane*, *Geometriae Dedicata* (to appear).
11. L. Santaló, *A theorem on sets of parallelepipeds with parallel edges*, *Publ. Inst. Mat. Univ. Nac. Litoral (Rosario)* 2 (1940), 49–60. (Spanish)
12. ———, *Supplement to the note: A theorem on sets of parallelepipeds with parallel edges*, *Publ. Inst. Mat. Univ. Nac. Litoral (Rosario)* 3 (1942), 202–210. (Spanish)
13. H. Rademacher and I. J. Schoenberg, *Helly's theorem on convex domains and Tchebycheff's approximation problem*, *Canad. J. Math.* 2 (1950), 245–256.

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