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ON UNIONS OF ν-EMBEDDED SETS

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ABSTRACT. Let A be a realcompact and C-embedded subspace of a space X and let B be a ν-embedded subspace of a space X. Then $A \cup B$ is ν-embedded in X.

1. Introduction. All spaces considered in this paper are assumed to be completely regular Hausdorff. For a space X, νX denotes the Hewitt realcompactification of X. A subset S of X is said to be C- (resp. C^*-) embedded in X if every real-valued continuous (resp. bounded continuous) function on S can be extended to a real-valued continuous function over X, and S is z-embedded in X if for each zero-set Z of S, there exists a zero-set Z' of X such that $Z = Z' \cap S$. Clearly, every C-embedded subset is C^*-embedded, and every C^*-embedded subset is z-embedded. In [1], R. L. Blair introduced the concept of ν-embedding as a generalization of z-embedding as follows. S is ν-embedded in X if the extension τ: $\nu S \to \nu X$ of the inclusion map i: $S \to X$ is a homeomorphism of νS into νX. Clearly, every realcompact subspace of X is ν-embedded in X. We denote that S is ν-embedded in X by $\nu S \subset \nu X$. In [1], R. L. Blair proved the following results.

(A) Assume that X is locally compact and that $S = (\bigcup_{i=1}^{n} A_i) \cup B$, where B is G_δ-closed and ν-embedded in X (so that $\nu B \subset \nu X$) and each A_i is realcompact and C-embedded in X. Then $\nu S = (\bigcup_{i=1}^{n} A_i) \cup \nu B$ (so S is ν-embedded in X).

(B) In any locally compact space X, the union of a compact set with a cozero-set is ν-embedded in X.

R. L. Blair asked whether in both cases above, the hypothesis of local compactness can be omitted. The purpose of this paper is to answer this question affirmatively. Furthermore, we shall show that the hypothesis of G_δ-closedness of B can be omitted.

Hereafter, $C(X)$ (resp. $C^*(X)$) denotes the set of all real-valued continuous (resp. bounded continuous) functions on a space X, N the space of natural numbers and R the space of real numbers. For realcompact spaces, the reader is referred to [1] and [2].

2. On unions of ν-embedded sets. Firstly we shall show the following lemma which is needed for our study.

LEMMA 2.1. *Let X be a space and A, B subspaces of X. If A is closed in νX and B is ν-embedded in X, then $A \cup B$ is C-embedded in $A \cup \nu B$.*

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PROOF. Let us denote $A \cup B$ by S. Let $f \in C(S)$. Then there exists a $g \in C(\nu B)$ which is the extension of $f|B$. Let us define a real-valued function $h: A \cup \nu B \to R$ as follows.

$$h(p) = \begin{cases} f(p) & \text{if } p \in A, \\ g(p) & \text{if } p \in \nu B. \end{cases}$$

Since $(A \cap \nu B) \cup B$ is C-embedded in νB, and B is dense in $(A \cap \nu B) \cup B$, $f|((A \cap \nu B) \cup B) = g|((A \cap \nu B) \cup B)$. So $f|(A \cap \nu B) = g|(A \cap \nu B)$. Hence the function h is well defined. In order to prove this lemma, it suffices to show that h is a continuous function on $A \cup \nu B$. If $p \in (A \cup \nu B) - A$, it is easy to prove that h is continuous at p. So we consider $p \in A$. Let G be an open neighborhood of $h(p)$ in R. Then there exists an open neighborhood H of $h(p)$ in R such that $\mathrm{cl}_R H \subset G$. Since $p \in A$, $h(p) = f(p)$. By the continuity of f, there exists an open neighborhood U of p in S such that $f(U) \subset H$. Then there exists an open set U^* in $A \cup \nu B$ such that $U^* \cap S = U$. Let $U_1^* = U^* \cap A$, $U_2^* = U^* \cap \nu B$ and $U_2 = U^* \cap B$. Then $U^* = U_1^* \cup U_2^*$ and $U_2^* \subset \mathrm{cl}_{\nu B} U_2$. So $h(U_1^*) = f(U_1^*) \subset G$ and

$$h(U_2^*) = g(U_2^*) \subset g(\mathrm{cl}_{\nu B} U_2) \subset \mathrm{cl}_R\, g(U_2) = \mathrm{cl}_R\, f(U_2) \subset \mathrm{cl}_R H \subset G.$$

Hence $h(U^*) = h(U_1^*) \cup h(U_2^*) \subset G$. So h is continuous at p. Hence h is a continuous function on $A \cup \nu B$, and the proof of Lemma 2.1 is completed.

COROLLARY 2.2. *Let X be a space and A, B subspaces of X. If B is ν-embedded in X and A is realcompact and C-embedded in X, then $A \cup B$ is C-embedded in $A \cup \nu B$.*

PROOF. Since every realcompact and C-embedded subset of X is closed in νX [2, 8.10(a)], this follows from Lemma 2.1.

THEOREM 2.3. *Let X be a space and A, B subspaces of X. If B is ν-embedded in X and A is realcompact and C-embedded in X, then $\nu(A \cup B) = A \cup \nu B$. So $A \cup B$ is ν-embedded in X.*

PROOF. By [1, Lemma 6.3], $A \cup \nu B$ is realcompact, and by Corollary 2.2, $A \cup B$ is C-embedded in $A \cup \nu B$. Since $A \cup B$ is dense in $A \cup \nu B$, $\nu(A \cup B) = A \cup \nu B$.

COROLLARY 2.4. *Let X be a space and $A_1, \ldots, A_n$, B subspaces of X. If B is ν-embedded in X and each A_i is realcompact and C-embedded in X, then $\nu((\bigcup_{i=1}^n A_i) \cup B) = (\bigcup_{i=1}^n A_i) \cup \nu B$. So $(\bigcup_{i=1}^n A_i) \cup B$ is ν-embedded in X.*

COROLLARY 2.5. *In any space X, the union of a compact subset with a ν-embedded subset is ν-embedded in X.*

Since every cozero-set of X is ν-embedded in X [1, Theorem 5.1], Corollary 2.4 and Corollary 2.5 are the answer to R. L. Blair's question quoted in the introduction. Next, we shall show that in Theorem 2.3 "C-embedded" cannot be weakened to "C^*-embedded". The following example is a modification of the Dieudonné plank in [3].

EXAMPLE 2.6. There exists a non-realcompact space which is the union of a realcompact subspace with a C^*-embedded Lindelöf subspace. Hence we cannot replace "C-embedded" by "C^*-embedded" in Theorem 2.3.

For any ordinal α, $W(\alpha)$ denotes the space of all ordinals less than α with the usual order topology. Let ω_1 be the first uncountable ordinal. Let D be the subset of $W(\omega_1)$ consisting of all isolated ordinals, and $\tilde{D} = D \cup \{\omega_1\}$. Then D is a discrete space of a nonmeasurable cardinal (so D is realcompact) and $\tilde{D}$ is the one-point Lindelöfication of D. Let βN be the Stone-Čech compactification of N. We construct X as follows.

$$X = \tilde{D} \times \beta N - (\{\omega_1\} \times (\beta N - N)).$$

We consider X as a subspace of the product space $\tilde{D} \times \beta N$. Let $A = D \times \beta N$ and $B = \{\omega_1\} \times N$. Then $X = A \cup B$ and A is realcompact and B is Lindelöf. The space X satisfies the following assertions.

Assertion 1. $\nu X = \tilde{D} \times \beta N$, so X is not realcompact.

Proof. Let $f \in C(X)$. Since each point of B is a P-point in X, we can choose a neighborhood U of ω_1 in $\tilde{D}$ such that f is constant on $U \times \{n\}$ for each $n \in N$. Since $U \times N$ is dense in $U \times \beta N - \{\omega_1\} \times (\beta N - N)$, f is constant on $(U - \{\omega_1\}) \times \{p\}$ for each $p \in \beta N - N$. Let p_U be the constant value of f on $(U - \{\omega_1\}) \times \{p\}$ for each $p \in \beta N - N$. We define a real-valued function $\tilde{f}$: $\tilde{D} \times \beta N \to R$ as follows.

$$\tilde{f}|X = f \quad \text{and} \quad \tilde{f}(\omega_1, p) = p_U \quad \text{for each } p \in \beta N - N.$$

Then $\tilde{f}$ is a continuous extension of f over $\tilde{D} \times \beta N$. Hence X is C-embedded in $\tilde{D} \times \beta N$. Since $\tilde{D} \times \beta N$ is realcompact and X is dense in $\tilde{D} \times \beta N$, $\nu X = \tilde{D} \times \beta N$.

Assertion 2. B is C^*-embedded in X.

Proof. Let $f \in C^*(B)$. Then there exists a $g \in C^*(\{\omega_1\} \times \beta N)$ such that $g|B = f$. Let us define a real-valued function h: $\tilde{D} \times \beta N \to R$ as follows.

$$h(\alpha, p) = g(\omega_1, p) \quad \text{for all } \alpha \in \tilde{D}, p \in \beta N.$$

Then $h|X$ is a continuous extension of f over X.

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