PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 3, July 1980

# ORTHOCOMPACTNESS AND PERFECT MAPPINGS

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ABSTRACT. An example is given which shows that orthocompactness is not preserved by perfect maps. Subparacompact pointwise star-orthocompact spaces are orthocompact; this shows that orthocompactness is preserved by closed maps in the presence of subparacompactness.

A space $X$ is said to be *orthocompact* if every open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ such that if $\mathcal{V}' \subset \mathcal{V}$, then $\bigcap \mathcal{V}'$ is open in $X$. Such a refinement $\mathcal{V}$ of $\mathcal{U}$ is called a *Q-refinement*, and any open collecton $\mathcal{W}$ such that $\bigcap \mathcal{W}'$ is open whenever $\mathcal{W}' \subset \mathcal{W}$ is called a *Q-collection*. The main purpose of this note is to provide an example showing the nonpreservation of orthocompactness under a perfect mapping, thus answering a question asked by B. Scott in **[S₁]** and **[S₂]**. The reader is referred to these papers for an in-depth discussion of orthocompactness, especially the product theory.

The description of the example follows below. We use the convention that an ordinal number is the set of smaller ordinals, and $I$ denotes the "closed unit interval" from $R$. A mapping is a continuous onto function.

EXAMPLE 1. There exists an orthocompact space $X$ and a perfect mapping $f$: $X \to Y$ onto a nonorthocompact space $Y$.

PROOF. Let $X_0 = \omega_1 \times I \times \{0\}$, $X_1 = \omega_1 \times I \times \{1\}$, and $X = X_0 \cup X_1$. For $\alpha, \beta \in \omega_1$, $\alpha$ a nonlimit ordinal with $\alpha \leq \beta$, $x \in I$, and $\varepsilon > 0$ define

$$B(\alpha, \beta, x, \varepsilon) = \{(\gamma, z, 0) \in X_0: \alpha \leq \gamma \leq \beta, 0 < |x - z| < \varepsilon\} \cup \{(\gamma, z, 1) \in X_1: \alpha \leq \gamma \leq \beta, |x - z| < \varepsilon\}.$$

Topologize $X$ by describing local bases as follows: Points $(\beta, x, 0) \in X_0$ are isolated in $X$. Points $(\beta, x, 1) \in X_1$ have the set of all $B(\alpha, \beta, x, \varepsilon)$, for nonlimit $\alpha \leq \beta$ and $\varepsilon > 0$, for a local base. It may be revealing to the reader to provide a simple sketch here, and realize that $X$ is similar to, but not quite the same as, the "Alexandroff double" of $\omega_1 \times I$.

To show $X$ is orthocompact, let $\mathcal{U}$ be an open cover of $X$. For each $x \in I$ consider $\mathcal{U}$ as an open cover of $H_x = \omega_1 \times \{x\} \times \{1\}$. There exists a nonlimit ordinal $\alpha_x < \omega_1$, an uncountable subset $A_x \subset [\alpha_x, \omega_1)$, and $\varepsilon_x > 0$ (use $\varepsilon_x = 1/n$ for some appropriate positive integer $n$) such that for each $\beta \in A_x$ we have $B(\alpha_x, \beta, x, \varepsilon_x) \subset U$ for some $U \in \mathcal{U}$. Note that the collection $\mathcal{W}_x = \{B(\alpha_x, \beta, x, \varepsilon_x): \beta \in A_x\}$ is a $Q$-collection. For $x \in I$, let $J_x = \{z \in I: |x - z| < \varepsilon_x\}$; then $\mathcal{J} = \{J_x: x \in I\}$ is an open cover of $I$ so there is a finite set $F \subset I$ such

Received by the editors July 2, 1979 and, in revised form, August 29, 1979.
*AMS (MOS) subject classifications* (1970). Primary 54D20; Secondary 54C10, 54G20.

that $I = \bigcup \{J_x : x \in F\}$. If $\beta_0 = \max\{\alpha_x : x \in F\}$, the subspace

$$Z = \{(\alpha, x, i) : 0 \leqslant \alpha \leqslant \beta_0, x \in I, i \in \{0, 1\}\}$$

is an open Lindelöf subspace of $X$, so there is an open cover $\mathcal{V}$ of $Z$ such that $\mathcal{V}$ is a $Q$-collection and each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. It follows that

$$\mathcal{W} = \mathcal{V} \cup (\bigcup \{\mathcal{W}_x : x \in F\}) \cup \{\{p\} : p \in X_0\}$$

is a $Q$-cover of $X$ that refines $\mathcal{U}$.

Now let $Y = X_0 \cup \omega_1$ and define a map $f\colon X \to Y$: $f(p) = p$ for $p \in X_0$ and $f(\alpha, x, 1) = \alpha$ for $(\alpha, x, 1) \in X_1$. Let $Y$ have the quotient topology induced by $f$. Clearly $f^{-1}(y)$ is compact in $X$ for each $y \in Y$ so to show $f$ is perfect it suffices to show $f$ is a closed mapping. Let $E \subset X$ be closed; to show $f(E)$ is closed in $Y$ we need only show that for any $\beta \in \omega_1 - f(E)$ there is an open neighborhood $V$ of $\beta$ in $Y$ such that $V \cap f(E) = \varnothing$. Now $f^{-1}(\beta) \cap E = (\{\beta\} \times I \times \{1\}) \cap E = \varnothing$ so for each $x \in I$ there is $\delta_x > 0$ and nonlimit $\alpha_x \leqslant \beta$ such that $B(\alpha_x, \beta, x, \delta_x) \cap E = \varnothing$. Using the compactness of $I$, we see that there is some $r_0 > 0$, nonlimit $\gamma_0 \leqslant \beta$, and a finite set $F \subset I$ such that

$$(\bigcup \{B(\gamma_0, \beta, x, r_0) : x \in F\}) \cap E = \varnothing$$

and $\{\beta\} \times I \times \{1\} \subset \bigcup \{B(\gamma_0, \beta, x, r_0) : x \in F\}$. Since $\bigcup \{B(\gamma_0, \beta, x, r_0) : x \in F\}$ is saturated with respect to $f$, we have $V = f(\bigcup \{B(\gamma_0, \beta, x, r_0) : x \in F\})$ as the desired neighborhood of $\beta$ in $Y$ where $V \cap f(E) = \varnothing$.

To see that $Y$ is not orthocompact, we note that if $\beta \in \omega_1$ and $U \subset Y$ is open, with $\beta \in U$, there must be some nonlimit $\alpha \leqslant \beta$ such that $[\alpha, \beta] \subset U$ and $[\alpha, \beta] \times \{x\} \times \{0\} \subset U$ for all but finitely many $x \in I$. Let $B = \{z_\alpha : \alpha < \omega_1\}$ be a subset of $I$, indexed by $\omega_1$, where $z_\alpha \neq z_\beta$ if $\alpha \neq \beta$. For each $\beta < \omega_1$ let $G_\beta = f(B(0, \beta, z_\beta, 1))$ and $\mathcal{G} = \{G_\beta : \beta < \omega_1\}$; then $\mathcal{G}$ is an open cover of $Y$ and if $\mathcal{H}$ is any open refinement of $\mathcal{G}$ there is some $\gamma \in \omega_1$ such that $[\gamma, \omega_1) \subset \mathrm{St}(\gamma, \mathcal{H})$. This can happen only if there is an uncountable set $A \subset [\gamma, \omega_1)$ where for each $\beta \in A$ there is $H_\beta \in \mathcal{H}$ such that $\gamma \in H_\beta \subset G_\beta$. It follows that

$$\left(\bigcap_{\beta \in A} H_\beta\right) \cap (\omega_1 \times \{z_\alpha\} \times \{0\}) = \varnothing$$

for every $\alpha \in A$, hence $\gamma \notin \mathrm{int}(\bigcap_{\beta \in A} H_\beta)$ and $\mathcal{H}$ cannot be a $Q$-refinement of $\mathcal{G}$. That concludes the verification of the stated properties of Example 1.

The proof, in the above example, that $Y$ is not orthocompact, was given for completeness. Other authors have considered similar examples and results which essentially show the nonorthocompactness of $Y$. G. Gruenhage **[G]** gave an example of a nonorthocompact space which is the closed image of an orthocompact space. Gruenhage's range space is homeomorphic to a closed subspace of $Y$ (and, under CH, is homeomorphic to $Y$) and certainly Gruenhage's result implies the nonorthocompactness of $Y$. The essential reason for the nonorthocompactness of $Y$ can also be culled from results in **[$S_1$]** or **[$S_2$]**, which show that $\omega_1 \times (\omega_1 + 1)$ is not orthocompact. For other related results on the construction of nonorthocompact spaces the reader is referred to **[HL]**.

The existence of Example 1 increases the importance of several generalizations of orthocompactness, considered by other authors, that are preserved under closed or perfect mappings. Weakly orthocompact spaces [$S_1$] are preserved under perfect maps, discretely orthocompact spaces [J] are preserved under closed mappings, and pointwise star-orthocompact spaces [G] are preserved under closed mappings. These concepts are useful in helping to preserve orthocompactness, under closed maps, when in the presence of other covering properties. Junnila [J] has shown that a $\theta$-refinable space $X$ is orthocompact if it is discretely orthocompact (see [J] for definition) and as a corollary he obtains:

THEOREM 2 [J]. *If $f: X \to Y$ is a closed continuous onto map, and $X$ is a $\theta$-refinable orthocompact space, then so is $Y$.*

A somewhat weaker result can be obtained via the pointwise star-orthocompactness defined by Gruenhage [G]. A space $X$ is *pointwise star-orthocompact* if for any open cover $\mathcal{U}$ of $X$ there is a $Q$-collection $\{V_x: x \in X\}$ such that $x \in V_x \subset \mathrm{St}(x, \mathcal{U})$ for each $x \in X$. Gruenhage shows that any pointwise star-orthocompact developable space is orthocompact; a modification of Gruenhage's proof actually yields the following stronger result.

THEOREM 3. *If $X$ is a subparacompact pointwise star-orthocompact space then $X$ is orthocompact.*

PROOF. If $\mathcal{U}$ is an open cover of the subparacompact space $X$ there is a sequence $\{\mathcal{G}_n\}_1^\infty$ of open covers of $X$ such that if $x \in X$ there is some $n \in N$ (depending on $x$) such that $\mathrm{St}(x, \mathcal{G}_n) \subset U$ for some $U \in \mathcal{U}$ (see [B]). Apply pointwise star-orthocompactness to each $\mathcal{G}_n$ and it follows that $\mathcal{U}$ has an open refinement which is the union of a countable number of $Q$-collections. Since a subparacompact space is countably metacompact, we see that $X$ is orthocompact [$S_1$].

Since subparacompactness is preserved under closed maps [B], we have the weaker version of Theorem 2, using "subparacompact" in place of "$\theta$-refinable".

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