

A COUNTABLY PARACOMPACT NONNORMAL SPACE

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ABSTRACT. A countably paracompact, first countable, separable, submetrizable, locally compact space which is not normal is constructed.

The theorem of M. Wage [7] which states a relationship between normal nonmetrizable Moore spaces and countably paracompact nonnormal Moore spaces has led to a search for first countable nonnormal countably paracompact spaces. Three examples have been constructed: one by Wage [6] which is separable submetrizable, one by van Douwen [2] which is separable and locally compact, and one by Vaughan [5] which is locally compact and even countably compact.

We now construct a locally compact, first countable, nonnormal countably paracompact space, $\langle X, \tau \rangle$, which is both submetrizable and separable. Since X is a subset of the real line and τ is finer than the Euclidean topology, by a theorem of [3], $\langle X, \tau \rangle$ is also real compact.

The construction of $\langle X, \tau \rangle$. Let \mathbf{Q} be the rationals, \mathbf{R} the reals and $c = |\mathbf{R}|$. We will define a topology τ on a subset $X \subset \mathbf{R}$ by inductively defining subsets $A_\beta = \{a_\alpha: \alpha < \beta\}$ and $B_\beta = \{b_\alpha: \alpha < \beta\}$ of \mathbf{R} and topologies τ_β on $X_\beta = A_\beta \cup B_\beta \cup \mathbf{Q}$, using techniques developed in [3] which originated in [4]. When we are finished with the inductive construction we will let $A = A_c$, $B = B_c$, $X = A \cup B \cup \mathbf{Q}$ and $\tau = \tau_c$.

We use the following inductive hypothesis I. H. (β) for $\beta < c$:

- (i) for all $\alpha < \beta$, $\mathcal{P}(X_\alpha) \cap \tau_\beta = \tau_\alpha$,
- (ii) τ_β refines the Euclidean topology on X_β ,
- (iii) \mathbf{Q} is dense in $\langle X_\beta, \tau_\beta \rangle$,
- (iv) $\langle X_\beta, \tau_\beta \rangle$ is locally compact,
- (v) $\langle X_\beta, \tau_\beta \rangle$ is locally countable (i.e. each point has a countable neighbourhood),
- (vi) A_β and B_β are closed in $\langle X_\beta, \tau_\beta \rangle$.

To begin, we let $A_0 = B_0 = \emptyset$ and τ_0 be the discrete topology on \mathbf{Q} . At limit stage λ , for $\lambda < c$, we let $A_\lambda = \cup \{A_\alpha: \alpha < \lambda\}$, $B_\lambda = \cup \{B_\alpha: \alpha < \lambda\}$ and τ_λ be the topology on X_λ generated by $\cup \{\tau_\alpha: \alpha < \lambda\}$. It is easy to check that the inductive hypotheses are fulfilled in these cases.

Before explaining the successor step we first let $\{C_\alpha^n = \langle C_\alpha^n: n \in \omega \rangle: \alpha < c\}$ enumerate all countable sequences of countable subsets of \mathbf{R} such that each sequence is repeated c times. Also, let $\{E_\alpha: \alpha < c\}$ enumerate all countable subsets of \mathbf{Q} .

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At stage $\beta + 1$ consider E_β and \mathcal{C}_β . Let $F_\beta = \{x \in \mathbf{R}: \text{for all Euclidean open neighbourhoods } N \text{ of } x, \text{ the set } \{n: N \cap C_\beta^n \neq \emptyset\} \text{ is infinite}\}$. We divide into two cases.

In the first case we suppose that $\bigcup \{C_\beta^n: n \in \omega\} \subseteq X_\beta$ and $|F_\beta| = c$. We pick $x \in F \setminus X_\beta$. There is a sequence $\{c_{n_k}: k \in \omega\}$ converging to x in the Euclidean topology such that each $c_{n_k} \in C_\beta^{n_k}$ and either the sequence is a subset of $A_\beta \cup \mathbf{Q}$ or $B_\beta \cup \mathbf{Q}$. Without loss of generality we assume $\{c_{n_k}: n \in \omega\} \subseteq A_\beta \cup \mathbf{Q}$. We now spilt into two subcases.

In the first subcase we suppose that x is a Euclidean limit point of E_β and that E_β is somewhere dense in the Euclidean topology. Pick $b \in \mathbf{R} \setminus (X_\beta \cup \{x\})$ such that b is a Euclidean limit point of E_β . Choose $\{y_k: k \in \omega\}$ and $\{z_k: k \in \omega\}$ disjoint subsets of E_β with $\{z_k: k \in \omega\} \cap \{c_{n_k}: k \in \omega\} = \emptyset$ such that $\{y_k: k \in \omega\}$ converges to x and $\{z_k: k \in \omega\}$ converges to b in the Euclidean topology. Now pick compact, clopen, countable neighbourhoods U_k of c_{n_k} such that $\{U_k: k \in \omega\}$ is discrete in τ_β with the diameter of $U_k < 1/k$. Also ensure that for each $k \in \omega$, $U_k \cap (B_\beta \cup \{z_i: i \in \omega\}) = \emptyset$. Now let $V_m^x = \bigcup \{U_k: k > m\} \cup \{y_k: k > m\}$ and $V_m^b = \{z_k: k > m\}$. Let $A_{\beta+1} = A_\beta \cup \{x\}$ and $B_{\beta+1} = B_\beta \cup \{b\}$ and $\tau_{\beta+1}$ be the topology generated by $\tau_\beta \cup \{V_m^x: m \in \omega\} \cup \{V_m^b: m \in \omega\}$.

For the second subcase, we assume that the first subcase does not happen. Hence x is a Euclidean limit point of $\mathbf{Q} \setminus E_\beta$ and $\mathbf{Q} \setminus E_\beta$ is somewhere dense. We now proceed in a manner similar to the first subcase, replacing E_β by $\mathbf{Q} \setminus E_\beta$.

For the second case, in which $\bigcup \{C_\beta^n: n \in \omega\} \not\subseteq X_\beta$ or $|F_\beta| < c$, we simply proceed as in the first case, ignoring all reference to the set F_β . Note that in any case either E_β has limit points in both $A_{\beta+1}$ and $B_{\beta+1}$ or $\mathbf{Q} \setminus E_\beta$ has limit points in both $A_{\beta+1}$ and $B_{\beta+1}$. We are now finished with the inductive construction.

The properties of $\langle X, \tau \rangle$. It is straightforward to check that $\langle X, \tau \rangle$ is locally compact, locally countable, separable and submetrizable. A and B are closed disjoint subsets of X which cannot be separated by open sets, since if $U \in \tau$, then for some β , $U \cap \mathbf{Q} = E_\beta$ and $X \setminus \bar{U} = \mathbf{Q} \setminus E_\beta$ at least one of which has limit points in both A and B .

To show that $\langle X, \tau \rangle$ is countably paracompact, it suffices to show that any countable open cover $\{U_n: n \in \omega\}$ with each $U_n \subseteq U_{n+1}$ has a locally finite refinement. Let $G_0 = U_0$ and $G_{n+1} = U_{n+1} \setminus U_n$. Let $\Phi = \{x \in X: \text{for all Euclidean open neighbourhoods } N \text{ of } x, \text{ the set } \{n \in \omega: G_n \cap N \neq \emptyset\} \text{ is infinite}\}$.

We claim that $|\Phi| < \aleph_0$. Suppose not. Let D be a countable subset of Φ which is somewhere dense for the Euclidean topology. For each $d \in D$ and for each $n \in \omega$ pick $g_n^d \in G_n$ such that in the τ topology d is in the closure of $\{g_n^d: n \in \omega\}$. Let $C^n = \{g_n^d: d \in D\}$ and $C = \{C^n: n \in \omega\}$.

Let $\Psi = \{x \in \mathbf{R}: \text{for each Euclidean open neighbourhood } N \text{ of } x \text{ the set } \{n \in \omega: N \cap C^n \neq \emptyset\} \text{ is infinite}\}$. Now Ψ is closed in the Euclidean topology, and contains D . Therefore $\Phi \subseteq \Psi$ and hence Ψ has cardinality c .

Now, there is some $\beta < c$ such that $C \subseteq X_\beta$ and $\langle C^n: n \in \omega \rangle = \mathcal{C}_\beta$ and hence by construction without loss of generality we can assume there is some $a \in A \cap \Psi$ (since otherwise there is $b \in B \cap \Psi$). Furthermore every τ neighbourhood of a

intersects infinitely many C^n 's and hence infinitely many G_n 's and therefore a is not in the τ interior of any U_n . This contradiction shows $|\Phi| < \aleph_0$.

Let $Y = X - \Phi$. $\{G_n: n \in \omega\}$ is locally finite for the Euclidean topology on Y . Hence there are Euclidean open sets $V_n \supseteq G_n$ such that $\{V_n: n \in \omega\}$ is locally finite for the Euclidean topology on Y .

Enumerate Φ as $\{x_n: n \in \omega\}$ and for each n pick a compact, countable $O_n \in \tau$ such that $x_n \in O_n$, diameter of $O_n < 1/n$ and $O_n \subset U_k$ for some k . Let $\Omega = \cup \{O_n: n \in \omega\}$. Ω is a clopen metrizable subset of $\langle X, \tau \rangle$.

Now, let $\{O_n^*: n \in \omega\}$ be a locally finite refinement of $\{O_n: n \in \omega\}$. Let $V_n^* = (V_n \setminus \Omega) \cap U_n$ for each $n \in \omega$. It is straightforward to check that $\{V_n^*: n \in \omega\} \cup \{O_n^*: n \in \omega\}$ refines $\{U_n: n \in \omega\}$.

REMARK. $\langle X, \tau \rangle$ is not paracompact, since it is not normal. In fact there is an open cover of cardinality \aleph_1 which has no point-countable refinement. Note that since $\langle X, \tau \rangle$ is separable any point-countable cover is countable.

Let $\{x_\alpha: \alpha \in c\}$ be an enumeration of X such that each initial segment is open and $Q = \{x_n: n \in \omega\}$. For each x_β find a countable neighbourhood U_β of x_β contained in $\{x_\alpha: \alpha < \beta\}$. Define a function $f: X \rightarrow \omega_1$ as follows:

$$f(x_\alpha) = \begin{cases} \alpha & \text{if } \alpha \in \omega_1, \\ \sup\{f(y): y \in U_\alpha\} & \text{if } \alpha \notin \omega_1. \end{cases}$$

Let $V_\alpha = \{x \in X: f(x) < \alpha\}$. $\{V_\alpha: \alpha \in \omega_1\}$ has no countable subcover.

Modification. We can modify the construction of $\langle X, \tau \rangle$ to obtain essentially the same space constructed in [1]. We eliminate all references to the sets A_β and B_β .

Let $\{\mathcal{P}_\alpha = \langle P_\alpha^1, P_\alpha^2 \rangle: \alpha \in c\}$ enumerate all pairs of countable subsets of \mathbb{R} such that each pair is repeated c times. We let our new inductive hypothesis be parts (i) – (v) of the previous inductive hypothesis.

The construction at limit stages is the same as before. At successor stage $\beta + 1$ we add a point $x_{\beta+1}$ ensuring that if P_β^1 and P_β^2 have c Euclidean limit points in common then $x_{\beta+1}$ is a limit point of both P_β^1 and P_β^2 in the new topology.

When we finish the construction the new space will have the property that if H and K are subsets of X and if H and K have uncountably many Euclidean limit points in common then H and K have c limit points in common in the new topology. We can use this property to show that the new space is normal. The proof is not dissimilar to the previous proof that $\langle X, \tau \rangle$ was countably paracompact and to the proof in [1]. Thus the new space will be separable, normal locally compact but with a cover of cardinality \aleph_1 having no point-countable refinement.

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