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SLICING CONVEX BODIES–BOUNDS FOR SLICE AREA IN TERMS OF THE BODY'S COVARIANCE

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ABSTRACT. Let Q be a zero-symmetric convex set in $\mathbf{R}^N$ with volume 1 and covariance matrix $V^2\mathrm{Id}_{N\times N}$. Let P be a K-dimensional vector subspace of $\mathbf{R}^n$, $K < N$, and let $J = N - K$. Then there exist constants $C_1(J)$ and $C_2(J)$ such that

$$V^{-J}C_1(J) < \mathrm{vol}_K(P \cap Q) < V^{-J}C_2(J).$$

The lower bound has applications to Diophantine equations.

1. Introduction. The restriction to bodies of covariance a constant multiple of $\mathrm{Id}_{N\times N}$ and volume 1 made in the abstract is not essential, as any centro-symmetric convex body can be brought to that form by a suitable linear transformation. Yet such bodies comprise the most important special cases. The unit cube, the L^1 ball $\Sigma_1^N|x_i| \leqslant r$ of volume 1 and the "complex cube" $|z_i| \leqslant \pi^{-1/2}$, $1 \leqslant i \leqslant N$, are examples.

For the case of the cube, real or complex, only the upper bound is of interest as there is a sharp lower bound of 1, independent of K and N, due to Vaaler **[10]**.[2] For the real cube in the case $K = N - 1$ an upper bound of 5 was given in **[6]**, which we improve here to $\sqrt{6}$.

Examples with a cube show that $C_1(J) \not> 12^{-J/2}$ and $C_2(J) \not< 6^{-J/2}$. We take

$$C_1(J) = (J+2)^{-J/2}\pi^{J/2}\Gamma\left(\tfrac{1}{2}J + 1\right),$$

$$C_2(J) = 2\left(8(\log 2)^{-J-3}(J+2)!\right)^{J/2}\pi^{J/2}\Gamma\left(\tfrac{1}{2}J+1\right) \quad \text{for } J \geqslant 2,$$

and $C_2(1) = 1/\sqrt{2}$. Then we have

THEOREM 1. *Let Q be a centro-symmetric convex body in $\mathbf{R}^N$ with volume U and diagonal covariance matrix $(V_i^2\delta_{ij})$, $1 \leqslant i, j \leqslant N$. Let P be a K-dimensional vector subspace of $\mathbf{R}^N$ with $K < N$, $K + J = N$. Let*

$$V = \left(U^{-N-2}\prod_1^N V_i^2\right)^{1/2N} \quad \text{and} \quad c_i = V_iV^{-1}U^{-1/2}.$$

Let W_K be the product of the K smallest c_i and W_K' the product of the K largest c_i.

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[2]The lower bound 1 for the case $K = N - 1$ is implicit in H. Hadwiger's *Gitterperiodische Punktmengen and Isoperimetrie*, Monatsh. Math. **76** (1972), 410–418.

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Then

$$C_1(J)W_KV^{-J} \leq \mathrm{vol}_K(P \cap Q) \leq C_2(J)W'_KV^{-J}.$$

If all the V_i are equal to some V, and $U = 1$ then all c_i, W_K, and W'_K are 1 and Theorem 1 reduces to

THEOREM 1′. *Let Q be a centro-symmetric convex body in $\mathbf{R}^N$ with volume* 1 *and covariance matrix $V^2\mathrm{Id}_{N\times N}$. Let P be a K-dimensional vector subspace of $\mathbf{R}^N$, $K < N$ and $J = N - K$. Then*

$$V^{-J}C_1(J) \leq \mathrm{vol}_K(P \cap Q) \leq V^{-J}C_2(J).$$

Theorem 1 follows from 1′ by an elementary lemma whose proof we omit. (But see Example 12, [**8**].)

LEMMA 1. *Let E be a convex set in $\mathbf{R}^N$ of dimension $K < N$. Let T be a linear transformation which maps each unit coordinate vector $\bar{e}_i$, $1 \leq i \leq N$, in $\mathbf{R}^N$ to $c_i\bar{e}_i$, with $c_i > 0$. Let $E' = TE$. Let W_K be the product of the K smallest c_i and W'_K the product of the K largest c_i. Then*

$$W_K\mathrm{vol}_K E \leq \mathrm{vol}_K E' \leq W'_K\mathrm{vol}_K E.$$

One may adapt our lower bound to Vaaler [**12**] by using a variant of his Lemma 6. Suppose $Q \subseteq \mathbf{R}^N$ has volume 1, is centro-symmetric and has covariance matrix $\mathrm{Cov}(Q) = V^2\mathrm{Id}_{N\times N}$. Let $L_j(\bar{x})$, $1 \leq j \leq N$, be N real linear forms in K variables, $L_j(\bar{x}) = \sum_{k=1}^K a_{jk}x_k$, so that $A = (a_{jk})$ is an $N \times K$ matrix, with $N > K$, $N = K + J$.

LEMMA 6′. *Let M be a positive integer and suppose that*

$$M|\det A^TA|^{1/2} \leq V^{-J}C_1(J).$$

Then there exist at least M distinct pairs of nonzero lattice points $\pm\bar{v}_m$, $1 \leq m \leq M$, such that for each m,

$$\bar{l}_m = \left(L_j(\bar{v}_m)\right) \in 2Q.$$

With Lemma 6′ in place of Lemma 6 of [**12**] and following [**12**] otherwise, we have a generalization of that paper's main result. Let $\Lambda_j(\bar{x}) = \sum_{k=1}^K a'_{jk}x_k$ for $1 \leq j \leq J$ be J real linear forms in K real variables $x_1, \ldots, x_K$. Let $N = J + K$. Let Q be centro-symmetric and convex with $\mathrm{vol}_N Q = 1$ and $\mathrm{Cov}(Q) = V^2\mathrm{Id}_{N\times N}$. For $\bar{y} \in \mathbf{R}^K$ let $l(\bar{y}) = (y_1, y_2, \ldots, y_K, \Lambda_1(\bar{y}), \ldots, \Lambda_J(\bar{y})) \in \mathbf{R}^N$.

THEOREM 2. *Let M be a positive integer and suppose that*

$$M^2\prod_1^J\left(1 + \sum_1^K |a'_{jk}|^2\right) \leq C_1(J)V^{-J}.$$

Then there exist M distinct pairs of nonzero lattice points $\pm\bar{v}_m$, $1 \leq m \leq M$, in $\mathbf{Z}^K$ such that $l(\pm\bar{v}_m) \in 2Q$.

REMARK. When Q is taken to be the unit cube we get a weaker version of Theorem 1 in [12] with all α_k and β_j there being 1; our more general lower bound is not as sharp for the cube as that in [11].

2. Proofs. Lemma 6′ follows from the proof in [11] of Theorem 2 (which appears in [12] as Lemma 6), but we use our Theorem 1′ in place of Theorem 1 of [11].

For the proof of Theorem 1 from Theorem 1′ we take the c_i in the statement of Theorem 1 as defining T, and let $Q' = T^{-1}Q$, $P' = T^{-1}P$. Applying Theorem 1′ to Q' and then Lemma 1 to the resulting bounds on $\mathrm{vol}_K(P \cap Q')$ gives Theorem 1.

It remains to prove Theorem 1′. For the proof we shall need some lemmas about centro-symmetric log concave functions. (A function f: $\mathbf{R}^J \to \mathbf{R}^+ \cup \{0\}$ is log concave if $\log f$ is concave.)

LEMMA 2. *Suppose f: $\mathbf{R}^J \to \mathbf{R}^+ \cup \{0\}$ satisfies $f(\bar{x}) = f(-\bar{x})$, $\{\bar{x}: f(\bar{x}) > t\}$ is convex and open for each t, and $\int_{\mathbf{R}^J} f(\bar{x})\, d^J(\bar{x}) = 1$. Suppose further that for all unit $\bar{\theta} \in \mathbf{R}^J$,*

$$\int_{\mathbf{R}^J} (\bar{x} \cdot \bar{\theta})^2 f(\bar{x})\, d^J(\bar{x}) \leqslant V^2.$$

Then

$$f(\bar{0}) \geqslant V^{-J}(J+2)^{-J/2}\pi^{-J/2}\Gamma\left(\tfrac{1}{2}J+1\right) = V^{-J}C_1(J).$$

PROOF. Let h: $\mathbf{R}^J \to \mathbf{R}^+ \cup \{0\}$ be constant at $V^{-J}C_1(J)$ for $\|\bar{x}\| < R = V(J+2)^{1/2}$, and 0 for $\|\bar{x}\| \geqslant R$. Then $\int_{\mathbf{R}^J} h = 1$ and $\int_{\mathbf{R}^J} x_i^2 h = V^2$, so h satisfies the hypotheses of Lemma 2. Thus to prove the lemma it suffices to prove that if $f \neq h$, $f(\bar{0}) \geqslant h(\bar{0})$.

We first show that f can be replaced by an f_1 such that f_1 is circular symmetric, that is, $f_1(\bar{x}) = f_1(\bar{y})$ if $\|\bar{x}\| = \|\bar{y}\|$, and such that $f_1(\bar{0}) = f(\bar{0})$ and f_1 satisfies the hypotheses of the lemma.

For let $E_t = \{\bar{x}: f(\bar{x}) > t\}$. By hypothesis E_t is convex, and if $t \leqslant s$, $E_s \subseteq E_t$. Thus

$$\int_{\mathbf{R}^J} f(\bar{x}) = \int_{t=0}^{f(0)} \int_{E_t} 1\, d^J(\bar{x})\, dt, \tag{3}$$

while

$$\int_{\mathbf{R}^J} x_i^2 f(\bar{x}) = \int_{t=0}^{f(0)} \int_{E_t} x_i^2\, d^J(\bar{x})\, dt. \tag{4}$$

Let E_t' be the ball about $\bar{0}$ of the same vol_J as E_t, and let $f_1(\bar{x}) = \sup\{t: \bar{x} \in E_t'\}$. Then

$$\int_{\mathbf{R}^J} f_1(\bar{x}) = \int_0^{f(0)} \int_{E_t'} 1\, d^J(\bar{x})\, dt = \int_0^{f(0)} \int_{E_t} 1\, d^J(\bar{x})\, dt = \int_{\mathbf{R}^J} f(\bar{x}) = 1. \tag{5}$$

Further,

$$\int_{\mathbf{R}^J} x_i^2 f_1(\bar{x}) = J^{-1}\int_{\mathbf{R}^J} \|\bar{x}\|^2 f_1(\bar{x}) = J^{-1}\int_0^{f(0)} \int_{E_t'} \|\bar{x}\|^2\, d^J(\bar{x})\, dt. \tag{6}$$

Now consider a particular value of t. If we show that $\int_{E_t'}\|\bar{x}\|^2 \leqslant \int_{E_t}\|\bar{x}\|^2$, the claim about f_1 is proved. So let INT $= E_t' \cap E_t$, EX $= E_t \setminus E_t'$ and EX$' = E_t' \setminus E_t$. Then

$$\int_{E_t}\|\bar{x}\|^2 = \int_{\mathrm{INT}}\|\bar{x}\|^2 + \int_{\mathrm{EX}}\|\bar{x}\|^2 \geqslant \int_{\mathrm{INT}}\|\bar{x}\|^2 + \int_{\mathrm{EX}'}\|\bar{x}\|^2 = \int_{E_t'}\|\bar{x}\|^2,$$

because $\|\bar{x}\|^2$ is everywhere in EX at least as large as anywhere in EX$'$ and $\mathrm{vol}_{J-1}\mathrm{EX} = \mathrm{vol}_{J-1}\mathrm{EX}'$. Thus $\int_{\mathbf{R}^J}\|\bar{x}\|^2 f_1(\bar{x})\, d^J(\bar{x}) \leqslant \int_{\mathbf{R}^J}\|\bar{x}\|^2 f(\bar{x})\, d^J(\bar{x})$, and because of the circular symmetry of f_1, $\int_{\mathbf{R}^J}(\bar{x}\cdot\bar{\theta})^2 f_1(\bar{x})\, d^J(\bar{x}) \leqslant V^2$ for all unit vectors $\bar{\theta}$.

Now suppose the lemma is false and $h(\bar{0}) > f(\bar{0}) = f_1(\bar{0})$. Since h and f_1 depend only on $\|\bar{x}\| = r$ we shall by an abuse of language write $h(r), f_1(r)$ for $h(\bar{x}), f_1(\bar{x})$ when $\|\bar{x}\| = r$. (For $r < R$, $h(r) = h(0)$, as well.) Now

$$\begin{aligned}\int_{\mathbf{R}^J} f_1(\bar{x})\, d^J(\bar{x}) &= \int_0^\infty f_1(r) J\sigma_J r^{J-1}\, dr = 1 = \int_{\mathbf{R}^J} h(\bar{x})\, d^J(\bar{x})\\ &= V^{-J}C_1(J)\cdot J\sigma_J \int_0^R r^{J-1}\, dr.\end{aligned}$$

Let $F_1(r) = \int_0^r f_1(u)J\sigma_J u^{J-1}\, du$ and $H(r) = \int_0^r h(u) J\sigma_J u^{J-1}\, du$. Since for $r < R$, $h(0) = h(r) > f_1(r)$ while for $r > R$, $0 = h(r) \leqslant f_1(r)$, we have $H(r) > F_1(r)$ for $0 < r < R$, and $H(r) \geqslant F_1(r)$ in any case. Thus

$$\int_0^\infty 2rF_1(r)\, dr < \int_0^\infty 2rH(r)\, dr$$

and

$$\int_0^\infty r^2\, dF_1(r) = \int_0^\infty r^2 f_1(r) J\sigma_J r^{J-1}\, dr > \int_0^\infty r^2\, dH(r) = \int_0^R r^2 h(0) J\sigma_J r^{J-1}\, dr.$$

In other words,

$$\int_{\mathbf{R}^J}\|\bar{x}\|^2 f_1(\bar{x})\, d^J(\bar{x}) > \int_{\mathbf{R}^J}\|\bar{x}\|^2 h(\bar{x})\, d^J(\bar{x}),$$

a contradicition. □

REMARK. Log concave centro-symmetric functions which satisfy the covariance hypothesis satisfy the other hypotheses of Lemma 2, as is proved in the preliminary lemma of [2]. Lemma 2 gives a sharp lower bound since an extremal function, h, is found. Our next lemma is not as sharp.

LEMMA 3. *Suppose* $f\colon \mathbf{R}^J \to \mathbf{R}^+ \cup \{0\}$ *satisfies* $f(\bar{x}) = f(-\bar{x})$, *is log concave, and* $\int_{\mathbf{R}^J} f(\bar{x})\, d^J(\bar{x}) = 1$. *Suppose further that for all unit* $\bar{\theta} \in \mathbf{R}^J$,

$$\int_{\mathbf{R}^J}(\bar{x}\cdot\bar{\theta})^2 f(\bar{x})\, d^J(\bar{x}) \geqslant V^2.$$

Then $f(\bar{0}) \leqslant V^{-J}C_2(J)$, *where* $C_2(J)$ *is the same as in Theorem* 1.

PROOF. Let $E_1 = \{\bar{x} \in \mathbf{R}^J\colon f(\bar{x}) > \frac{1}{2}f(\bar{0})\}$. E_1 is convex and centro-symmetric. Let $E_i = iE_1 \setminus (i-1)E_1$ for $i > 1$, so that $\mathbf{R}^J$ is the disjoint $\bigcup_{i=1}^\infty E_i$. Let r be the minimal radius of E_1, and let $\bar{\theta}$ be a unit vector in the direction of a point on ∂E_1 of norm r.

Since $\int_{\mathbf{R}^J} (\bar{x}\cdot\bar{\theta})^2 f(\bar{x})\, d^J(\bar{x}) \geqslant V^2$,

$$V^2 \leqslant \sum_{i=1}^{\infty} \int_{E_i} (\bar{x}\cdot\bar{\theta})^2 f(\bar{x})\, d^J(\bar{x}) \leqslant 4\sum_{i=1}^{\infty} i^{J+2}2^{-i}r^2$$

because $\mathrm{vol}_J E_1 \leqslant 2/f(\bar{0})$ and $\int_{E_1}(\bar{x}\cdot\bar{\theta})^2 f(\bar{x})\, d^J(\bar{x}) \leqslant (2/f(\bar{0}))r^2 f(\bar{0}) = 2r^2$. Now

$$4\sum_{1}^{\infty} i^{J+2}2^{-i}r^2 \leqslant 8r^2\int_0^{\infty} s^{J+2}2^{-s}\, ds = Dr^2 = 8r^2(\log 2)^{-J-3}(J+2)!$$

by the integral comparison test. Thus $r \geqslant D^{-1/2}V$. On the other hand $2 \geqslant f(\bar{0})\mathrm{vol}_J(E_1) \geqslant f(\bar{0})\sigma_J r^J$ so

$$\begin{aligned} f(\bar{0}) &\leqslant 2r^{-J}\sigma_J^{-1} \leqslant 2V^{-J}D^{J/2}\sigma_J^{-1} \\ &= V^{-J}\cdot 2\big(8(\log 2)^{-J-3}(J+2)!\big)^{J/2}\pi^{-J/2}\Gamma\big(\tfrac{1}{2}J+1\big) = V^{-J}C_2(J). \quad \square \end{aligned}$$

In case $J = 1$ we can find the extremal function and obtain a sharper upper bound.

LEMMA 4. *Suppose f: $\mathbf{R}\to\mathbf{R}^+ \cup \{0\}$ satisfies $f(x) = f(-x)$, is log concave, and $\int_{\mathbf{R}} f(x)\,dx = 1$, $\int_{\mathbf{R}} x^2 f(x)\,dx = V^2$. Then $f(0) \geqslant 2^{-1/2}V^{-1}$.*

PROOF. Let $\alpha = f(0)$. Since $\int_{-\infty}^{\infty} e^{-2\alpha|x|}\,dx = 1$, $f(x)$ cannot be everywhere $\geqslant \alpha e^{-2\alpha|x|}$. They are equal at 0, so either $f(x) \equiv \alpha e^{-2\alpha|x|}$ or there exists $\beta > 0$ such that for $0 < x < \beta$, $f(x) > \alpha e^{-2\alpha|x|}$ while for $x > \beta$, $f(x) < \alpha e^{-2\alpha|x|}$. (Both are log concave.)

Now let $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_0^x \alpha e^{-2\alpha|t|}\,d$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} x^2(f(x) - \alpha e^{-2\alpha|x|})\,dx &= 2\int_0^{\infty} x^2(f(x) - \alpha e^{-2\alpha x})\,dx \\ &= 4\int_0^{\infty} x(G(x) - F(x))\,dx \end{aligned}$$

since $x^2(F - G)$ declines exponentially to zero at $\pm\infty$. Since $F(0) = G(0) = 0$ and $F(\infty) = G(\infty) = \frac{1}{2}$, and since $d(F - G)/dx > 0$ only for $0 < x < \beta$, $x(G(x) - F(x))$ is always negative and $\int_{-\infty}^{\infty} x^2(f(x) - \alpha e^{-2\alpha|x|})\,dx < 0$. $\square$

What does all this have to do with $\mathrm{vol}_K(P \cap Q)$? Let K_J denote the unit cube $|x_i| \leqslant \frac{1}{2}$, $K < i \leqslant N$. Let B denote the null matrix of P, so that B has N columns and J rows, and $B\bar{x} = \bar{0}$ if $\bar{x} \in P$. Without loss of generality we may assume $\mathrm{rank}(B) = J$. Let $B_\varepsilon = [B|\varepsilon\mathrm{Id}_{J\times J}]$, and let P_ε be the null space of B_ε. Then

$$\lim_{\varepsilon\to 0} \mathrm{vol}_N(P_\varepsilon \cap (Q \times K_J)) = \mathrm{vol}_K(P \cap Q), \tag{7}$$

and

$$\mathrm{vol}_N(P_\varepsilon \cap (Q \times K_J)) = \mathrm{vol}_N\{\bar{z}\colon B_\varepsilon\bar{z} = 0 \text{ and } \bar{z} \in Q \times K_J\}. \tag{8}$$

We wish to compare the volume in (8) to its projection onto the first N coordinates, $\mathrm{vol}_N(\{\bar{x} \in Q\colon B\bar{x} \in \varepsilon K_J\})$. This ratio is $\varepsilon^{-J}(\det B_\varepsilon B_\varepsilon^T)^{1/2}$, which we now prove.

We shall need a lemma about how areas are affected by projection. If $\bar{a}_1 \ldots \bar{a}_N$ are column vectors in $\mathbf{R}^{N+J}$, if $A = [\bar{a}_1\bar{a}_2 \ldots \bar{a}_N]^T$ is a matrix of $N + J$ columns and N rows, and $\text{Box}_A = \{\bar{a}: \bar{a} = \sum_1^N \lambda_i \bar{a}_i, 0 \leq \lambda_i \leq 1\}$ then $\text{vol}_N \text{Box}_A = (\det AA^T)^{1/2}$ (Eves [3]).

Now suppose A has the form $[\text{Id}_{N \times N}| - D]$ and let $A' = [D^T|\text{Id}_{J \times J}]$.

LEMMA 5. $\text{Det}(AA^T) = \text{Det}(A'A'^T)$.

PROOF. By the Cauchy-Binet theorem, $\text{Det}(AA^T) = \sum_\alpha (\det \alpha)^2$ where the summation is over all $N \times N$ minors α of A [5]. Now a typical α consists of l columns of $\text{Id}_{N \times N}$ and m columns of D, $l \leq N$, $m \leq J$, $l + m = N$.

$$\alpha = \left(\bar{e}_{\alpha_1}\bar{e}_{\alpha_2} \cdots \bar{e}_{\alpha_l} - \bar{D}_{\alpha_{1+l}} - \bar{D}_{\alpha_{2+l}} - \cdots - \bar{D}_{\alpha_{m+l}}\right).$$

Expanding $\det \alpha$ about the 1st through lth columns, each of which contain 1 once and otherwise zeros, we have $\text{Det}(\alpha) = \pm \text{Det}(\beta)$ where β is the square matrix consisting of the intersection of the $N - l$ rows not indexed by $\alpha_1, \alpha_2 \ldots \alpha_l$ and the m columns of $-D$ belonging to α. Thus $\det(AA^T) = 1 + \sum_\beta (\det \beta)^2$, the sum taken over all $m \times m$ minors of D, $m \leq J$. But $\text{Det}(A'A'^T) = \sum_\gamma (\det \gamma)^2$, sum over $J \times J$ minors γ of A', and these may be expanded about the columns of $\text{Id}_{J \times J}$ to obtain again $1 + \sum_\beta (\det \beta)^2$. □

Now let $\bar{b}_i$, $1 \leq i \leq N$, denote the column vectors of B. Let

$$\bar{a}_i = \left[\frac{\bar{e}_i}{-\varepsilon^{-1}\bar{b}_i}\right] \quad \text{and} \quad A^T = [\bar{a}_1 \bar{a}_2 \ldots \bar{a}_N].$$

Then $B_\varepsilon A^T = (0_{J \times N})$ so that the $\bar{a}_i$ are vectors in $\text{Null}(B_\varepsilon)$. When these $\bar{a}_i$ are projected onto their first N coordinates they are $\bar{e}_1, \bar{e}_2 \ldots \bar{e}_N$. Therefore they are linearly independent and form a basis of $\text{Null}(B_\varepsilon) = P_\varepsilon$.

Now $\text{vol}_N \text{Box}_A = (\det AA^T)^{1/2}$ while $\text{vol}_N \text{Proj}(\text{Box}_A) = 1$ as $\bar{e}_1 \ldots \bar{e}_N$ are orthonormal. The ratio of volumes, $(\det AA^T)^{1/2}$, is independent of which measurable set is projected.

Now A has the form $A = [\text{Id}_{N \times N}| - D]$ with $D = \varepsilon^{-1}B^T$, so that $A' = [\varepsilon^{-1}B|\text{Id}_{J \times J}]$. Thus

$$\begin{aligned}(\det AA^T)^{1/2} &= (\det A'A'^T)^{1/2} \quad \text{by Lemma 5} \\ &= \varepsilon^{-J}(\det B_\varepsilon B_\varepsilon^T)^{1/2},\end{aligned}$$

and this is the ratio of a volume in P_ε to the volume of its projection onto the first N coordinates, as claimed.

Since we can find $\text{vol}_N(P_\varepsilon \cap (Q \times K_J))$ from $\text{vol}_N\{\bar{x} \in Q: B\bar{x} \in \varepsilon K_J\}$, we turn our attention to this last. It may be regarded as the probability that an $\bar{x}$ taken "at random" from Q (the probability measure being Lebesgue measure restricted to Q) will satisfy $B\bar{x} \in \varepsilon K_J$. Let $f(\bar{x})$ denote the probability density function of $B\bar{x}$. Since Q is convex, $f(\bar{x})$ is log concave. (This is the key observation.)

For let μ denote the probability measure associated with f, and ν Lebesgue measure restricted to Q. Let $s = 1 - t$, $0 \leq t \leq 1$, and let C, D be open convex sets in $\mathbf{R}^J$. Let $B^{-1}C$, $B^{-1}D$ be the inverse images in $\mathbf{R}^N$ under B of C and D respectively. From Prekópa [9], since χ_Q, the characteristic *function* of Q, is log concave, the measure ν is log concave, that is, $\nu(sC' + tD') \geq (\nu(C'))^s(\nu(D'))^t$. Let

$C' = B^{-1}D$, $D' = B^{-1}D$. Then

$$\mu(sC + tD) = \nu(B^{-1}(sC + tD)) = \nu(sB^{-1}C + tB^{-1}D) \quad \text{(since } B \text{ is linear)}$$
$$= \nu(sC' + tD') > (\nu(C'))^s(\nu(D'))^t = (\mu(C))^s(\mu(D))^t.$$

So μ is a log concave measure and, again by Prekópa [9], f is a log concave function. Now as $\varepsilon \to 0$,

$$\mathrm{Prob}(B\bar{x} \in \varepsilon K_J) \sim \varepsilon^J f(\bar{0}). \tag{9}$$

Also,

$$\mathrm{Cov}(f)_{ij} = \int_{\mathbf{R}^J} x_i x_j f(\bar{x})\, d^J(\bar{x}) = (V^2 BB^T)_{ij}. \tag{10}$$

Let $\bar{X}$ be the random vector uniformly distributed on Q. Since BB^T is selfadjoint and positive definite (rank $B = J$), there is a square matrix S such that $SS^T = BB^T$ [1].

Let Y be the random vector $\bar{Y} = S^{-1}B\bar{X}$. Then $\mathrm{Cov}(B\bar{X}) = V^2BB^T$, so $\mathrm{Cov}(\bar{Y}) = V^2S^{-1}BB^TS^{-1T} = V^2\mathrm{Id}_{J\times J}$, since for any random vector $\bar{Z}$, any square matrix U, $\mathrm{Cov}(U\bar{Z}) = E(U\bar{Z}(U\bar{Z})^T) = E(U\bar{Z}\bar{Z}^TU^T) = UE(\bar{Z}\bar{Z}^T)U^T = U\,\mathrm{Cov}(\bar{Z})U^T$ because expectation (E) is linear.

Let $h(\bar{x})$ be the probability density function associated with $\bar{Y}$. Then

$$h(\bar{0}) = (\det BB^T)^{1/2} f(\bar{0}), \tag{11}$$

$$\mathrm{Cov}(h) = \mathrm{Cov}(\bar{Y}) = V^2\mathrm{Id}_{J\times J}, \tag{12}$$

and since log concavity is preserved under the linear transformation S^{-1}, h is log concave.

Applying Lemmas 2 and 3 to h, and Lemma 4 in case $J = 1$, we have from (12) that

$$C_1(J)V^{-J} < h(\bar{0}) < C_2(J)V^{-J} \tag{13}$$

and from (7) through (11) that

$$C_1(J)V^{-J} < \mathrm{vol}_K(P \cap Q) < C_2(J)V^{-J}. \quad \square$$

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