NONEXISTENCE OF COREFLECTIONS AND COPRODUCTS IN SOME SUBCATEGORIES OF SEMIGROUPS

P. G. TROTTER

ABSTRACT. It is shown here that there are no nontrivial free regular semigroups, free regular semigroups on a semigroup or free products of regular semigroups. Analogous results are obtained in the categories of simple, bisimple and bisimple regular semigroups.

1. Introduction. Let C denote a subcategory of the category of semigroups. Denote by $C(X)$ a coreflection object in C for the set X , by $C(S)$ a coreflection object in C for a semigroup S and by $C(\chi)$ a coproduct object for the family $\chi = \{S_i; i \in I\}$ in C [4]. In accordance with usual terminology these are respectively a free C -semigroup, a free C -semigroup on a semigroup and a free product in C . Call $C(X)$, $C(S)$ or $C(\chi)$ *trivial* if $X = \square$, S is an object of C or $|I| < 1$ respectively.

There are subcategories C of semigroups in which these objects exist for any sets, semigroups and families in C . In particular they exist in the categories of semigroups and inverse semigroups [1], [6], [3], [5]. It is proved in this paper, for C the category of regular semigroups, that there are no nontrivial objects $C(X)$, $C(S)$ or $C(\chi)$. Analogous results for the categories of simple, bisimple and bisimple regular semigroups are provable but since the proofs in these cases are essentially similar to that for regular semigroups, they are not presented in detail.

2. The construction. Let S be a semigroup. In this section a regular semigroup $\bar{S}$ is constructed so that $\bar{S}$ has regular subsemigroups R and T with $R \cap T = S$. With this construction the claimed results can be proved. The construction is based on that described in [3].

Choose a set of distinct elements $Z_1 = \{x_a, y_a; a \in S\}$ so that $Z_1 \cap S = \square$. Let $X_1 = \{x_a; a \in S\}$ and $Y_1 = \{y_a; a \in S\}$. We have $X_1 \cap Y_1 = \square$. Denote by $F(Z_1)$ the free semigroup on Z_1 . Let $F_1 = S * F(Z_1)$ be the free product of S and $F(Z_1)$. Put $H_1 = S * F(X_1)$ and $K_1 = S * F(Y_1)$. Let τ_1 be the symmetric relation on F_1 generated by $\{(ax_aa, a), (ay_aa, a); a \in S\}$ and let ρ_1 be the least congruence on F_1 that contains τ_1 . Write $S_1 = F_1/\rho_1$, $R_1 = H_1/\rho_1$ and $T_1 = K_1/\rho_1$.

The aim initially is to prove that $R_1 \cap T_1 = (S)\rho_1^* \simeq S$. Some definitions and results are required for this.

Note that $(u, v) \in \rho_1$ if and only if there exists $p_i, q_i \in F_1^1$ and $(u_i, v_i) \in \tau_1$ for $1 \leq i \leq n$ so that

$$u = p_1 u_1 q_1, p_1 v_1 q_1 = p_2 u_2 q_2, \dots, p_n v_n q_n = v. \quad (1)$$

Received by the editors February 16, 1978 and, in revised form, June 11, 1979.

AMS (MOS) subject classifications (1970). Primary 20M10. © 1980 American Mathematical Society
0002-9939/80/0000-0401/\$02.25

If at step i in this chain $u_i = a$ and $v_i = az_a a$ we call the step and z_a an *insertion*. If $u_i = az_a a$ and $v_i = a$ call the step and z_a a *deletion*. A term $z_a \in Z_1$ may appear several times in the free product expression for u . Index these occurrences and the occurrences of z_a as insertions in the chain (1). If the indexed term $(z_a)_m$ appears as both an insertion and a deletion in (1), we say z_a is an *insertion-deletion* in (1).

LEMMA 2.1. *If $(u, v) \in \rho_1$ then there is a chain of form (1) for u, v with no insertion-deletions.*

PROOF. Suppose there are insertion-deletions in (1). Choose i, j so that $(z_a)_m$ is inserted at step i and deleted at step j and so that no insertion-deletions occur between steps i and j . Call $(z_a)_m$ a *central insertion-deletion*. Then $u_i = a \in S$, $v_i = az_a a$, $u_j = az_a a$ and $v_j = a$. From the definition of F_1 as a free product we have $(p_i a, p_j a), (aq_i, aq_j) \in \rho_1$. Suppose $p_i a$ is changed to f at some step between i and j . If f is obtained by a deletion then $f \in F_1^1 a$. Since there are no insertion-deletions between steps i and j , if f is obtained by an insertion then also $f \in F_1^1 a$. Dual statements apply for changes in aq_i . Hence, for $i < h < j$ and some $f_h, g_h \in F_1^1$, it can be seen that either $p_h = f_h a(z_a)_m g_h$ where $g_h u_h q_h, g_h v_h q_h \in aF_1^1$ or $q_h = f_h (z_a)_m a g_h$ where $p_h u_h f_h, p_h v_h f_h \in F_1^1 a$. Define respectively either $\bar{p}_h = f_h g_h$ and $\bar{q}_h = q_h$ or $\bar{p}_h = p_h$ and $\bar{q}_h = f_h g_h$. Then

$$\begin{aligned} u &= p_1 u_1 q_1, \dots, p_{i-1} v_{i-1} q_{i-1} = \bar{p}_{i+1} u_{i+1} \bar{q}_{i+1}, \dots, \bar{p}_{j-1} v_{j-1} \bar{q}_{j-1} \\ &= p_{j+1} u_{j+1} q_{j+1}, \dots, p_n v_n q_n = v \end{aligned}$$

is a chain of form (1) which has less insertion-deletions than (1). So the lemma can be proved by induction.

REMARK 2.2. Note that by this lemma it follows that $S \simeq (S)\rho_1^\#$.

LEMMA 2.3. *Suppose $u \in H_1$, $v \in K_1$ and $(u, v) \in \rho_1$. There is a chain of form (1) for u, v with no insertion-deletions and so that deletions from $F(X_1)$ precede insertions from $F(Y_1)$.*

PROOF. Assume, by Lemma 2.1, that (1) has no insertion-deletions. Hence (1) involves only deletions from $F(X_1)$ and insertions from $F(Y_1)$. Suppose $u_i = b$, $v_i = b(y_b)_h b$, $u_{i+1} = a(x_a)_k a$ and $v_{i+1} = a$ where $y_b \in F(Y_1)$ and $x_a \in F(X_1)$. Let $w = p_i v_i q_i = p_{i+1} u_{i+1} q_{i+1}$. Assume, in the free product expression for w , that $(x_a)_k$ precedes $(y_b)_h$; a similar argument applies otherwise. Then $w = p_{i+1} a(x_a)_k e(y_b)_h b q_i$ where $e = af = gb$ and $p_i = p_{i+1} a(x_a)_k g$, $q_{i+1} = f(y_b)_h b q_i$, for some $e, f, g \in F_1^1$. We get

$$p_i b q_i = p_{i+1} a(x_a)_k a f q_i, \quad p_{i+1} a f q_i = p_{i+1} g b q_i, \quad p_{i+1} g b(y_b)_h b q_i = p_{i+1} a q_{i+1}.$$

Defining $\bar{p}_i = p_{i+1}$, $\bar{q}_i = f q_i$, $\bar{p}_{i+1} = p_{i+1} g$, $\bar{q}_{i+1} = q_i$, $\bar{u}_i = u_{i+1}$, $\bar{v}_i = v_{i+1}$, $\bar{u}_{i+1} = u_i$ and $\bar{v}_{i+1} = v_i$ then

$$u = p_1 u_1 q_1, \dots, p_{i-1} v_{i-1} q_{i-1} = \bar{p}_i \bar{u}_i \bar{q}_i,$$

$$\bar{p}_i \bar{v}_i \bar{q}_i = \bar{p}_{i+1} \bar{u}_{i+1} \bar{q}_{i+1}, \quad \bar{p}_{i+1} \bar{v}_{i+1} \bar{q}_{i+1} = p_{i+2} u_{i+2} q_{i+2}, \dots, p_n v_n q_n = v,$$

which is a chain of form (1) with less inclusions from $F(Y_1)$ preceding deletions from $F(X_1)$ than (1) has. Hence the lemma can be proved by induction.

REMARK 2.4. By Lemma 2.3, if $u \in H_1$, $v \in K_1$ and $(u, v) \in \rho_1$ then there exists $s \in S$ so that $(u, s) \in \rho_1$. Hence $R_1 \cap T_1 = (S)\rho_1^\#$.

Note that S_1 , R_1 and T_1 are not regular semigroups. The desired semigroups $\bar{S}$, R and T will be constructed inductively to be regular "closures" of S_1 , R_1 and T_1 respectively.

For the sake of the next step of the construction define an endomorphism $\alpha_1: F_1 \rightarrow F_1$ by $aa_1 = a$ and $y_a\alpha_1 = x_a = x_a\alpha_1$ for each $a \in S$, $x_a \in X_1$ and $y_a \in Y_1$. Note that if $(u, v) \in \rho_1$ then the image under α_1 of a chain of form (1) for u and v is a chain of form (1) for $(u)\alpha_1$ and $(v)\alpha_1$. Hence by the induced homomorphism theorem [1] there is a homomorphism $\beta_1: S_1 \rightarrow R_1$ so that $\rho_1^\#\beta_1 = \alpha_1\rho_1^\#$. Since α_1 is the identity map on H_1 then β_1 is the identity map on R_1 .

The construction can now be continued. Define a set of distinct elements $Z_2 = \{z_a; a \in S_1\}$ so that $Z_2 \cap S_1 = \emptyset$. Let $X_2 = \{z_a \in Z_2; a \in R_1\}$ and $Y_2 = \{z_a \in Z_2; a \in T_1\}$. Define $F_2 = S_1 * F(Z_2)$, $H_2 = R_1 * F(X_2)$ and $K_2 = T_1 * F(Y_2)$. Let τ_2 be the symmetric relation on F_2 generated by $\{(az_a a, a); a \in S_1\}$ and let ρ_2 be the least congruence on F_2 containing τ_2 . Put $S_2 = F_2/\rho_2$, $R_2 = H_2/\rho_2$ and $T_2 = K_2/\rho_2$.

It will now be shown that $R_2 \cap T_2 = (S)\rho_1^\#\rho_2^\# \simeq S$.

Lemmas 2.1 and 2.3 can be proved as above if in their statements ρ_2 , H_2 , K_2 , X_2 and Y_2 replace ρ_1 , H_1 , K_1 , X_1 and Y_1 respectively. So $(S_1)\rho_2^\# \simeq S_1$ and in particular, by Remark 2.2, $(S)\rho_1^\#\rho_2^\# \simeq S$.

Suppose $u \in H_2$, $v \in K_2$ and $(u, v) \in \rho_2$. By the generalisation of Lemma 2.3 and Remark 2.4, there exists $s \in S_1$ so that $(u, s) \in \rho_2$, and there is a chain of form (1) for u and s involving only deletions from $F(X_2)$. Let $\alpha_2: F_2 \rightarrow F_2$ be the endomorphism defined by $aa_2 = b$, $z_a\alpha_2 = z_b$ for any $z_a \in F_2$, $a \in S_1$ and $b = a\beta_1$. Then α_2 is the identity map on H_2 . By considering the image under α_2 of the chain (1) for u and s we see that $u = u\alpha_2 \rho_2 s\alpha_2 \in R_1$. Likewise $v\rho_2^\# \in (T_1)\rho_2^\#$. Since $(S_1)\rho_2^\# \simeq S_1$ then $R_2 \cap T_2 = (R_1 \cap T_1)\rho_2^\# \simeq S$.

Let $\beta_2: S_2 \rightarrow R_2$ be the induced homomorphism so that $\rho_2^\#\beta_2 = \alpha_2\rho_2^\#$.

Since $\rho_1^\#$ and $\rho_2^\#$ restricted to S and S_1 respectively are injective homomorphisms, assume for convenience that $S \subseteq S_1 \subseteq S_2$. We may similarly construct S_3 , R_3 and T_3 so that $S_2 \subseteq S_3$, $R_3 \cap T_3 = S$ and each element of S_2 , R_2 and T_2 has an inverse in S_3 , R_3 and T_3 respectively. Continuing by induction, semigroups S_n , R_n and T_n can be constructed so that $S_{n-1} \subseteq S_n$, $R_n \cap T_n = S$ and each element of S_{n-1} , R_{n-1} and T_{n-1} has an inverse in S_n , R_n and T_n respectively.

To complete the construction let $\bar{S} = \bigcup_{i=1}^\infty S_i$, $R = \bigcup_{i=1}^\infty R_i$ and $T = \bigcup_{i=1}^\infty T_i$. These semigroups are regular, $R \subseteq \bar{S} \supseteq T$ and $R \cap T = S$.

THEOREM 2.5. *If S is a nonregular semigroup then there exists no free regular semigroup on S .*

PROOF. Suppose F is a free regular semigroup on S . Then there is a homomorphism $\alpha: S \rightarrow F$ so that for any homomorphism $\beta: S \rightarrow V$, V a regular semigroup, there is a unique homomorphism $\gamma: F \rightarrow V$ where $\alpha\gamma = \beta$. By the construction S is embeddable in a regular semigroup $\bar{S}$ with regular subsemigroups R and T so that

$R \cap T \simeq S$. Let $V = \bar{S}$ and $(S)\beta = R \cap T$. Since $(F)\gamma$ is regular, while S is not, there are at least two distinct choices for the map γ ; namely with $(F)\gamma \subseteq R$ or $(F)\gamma \subseteq T$.

COROLLARY 2.6. *There are no nontrivial free regular semigroups.*

PROOF. If there is a free regular semigroup on a set X then it can be readily seen that there is a free regular semigroup on the free semigroup $F(X)$. $F(X)$ is nonregular.

COROLLARY 2.7. *There are no nontrivial free products in the category of regular semigroups.*

PROOF. Let F be a free product in the category of semigroups of the family $\{S_i; i \in I\}$ of regular semigroups. It can be easily checked that a free product of this family in the category of regular semigroups exists if and only if the free regular semigroup on F exists.

3. Some analogous cases. If C is the category of simple, bisimple or bisimple regular semigroups the construction of §2 can be remodelled to give analogous results. These categories are defined by existential conditions, as is the category of regular semigroups. For example, a semigroup B is bisimple if and only if for each $a, b \in B$ there exists $x_{ab}, y_{ab}, u_{ab}, v_{ab} \in B$ so that $x_{ab}bu_{ab} = a, y_{ab}av_{ab} = b$ and $bu_{ab} = y_{ab}a$.

For simple semigroups the construction can be readily modified. However, since the bisimple case is more complicated, a brief outline is presented here.

Let S be a semigroup and X_1 be a set of distinct elements $X_1 = \{x_{ab}, y_{ab}, u_{ab}, v_{ab}; (a, b) \in S \times S\}$ where $X_1 \cap S = \emptyset$. Define Y_1 similarly so that $X_1 \cap Y_1 = \emptyset$ and let $Z_1 = X_1 \cup Y_1$. Define F_1, H_1 and K_1 as in §2. Let τ_1 be the symmetric relation on F_1 generated by the subset $\{(x_{ab}bu_{ab}, a), (y_{ab}av_{ab}, b), (bu_{ab}, y_{ab}a); (a, b) \in S \times S\}$ of H_1 and the corresponding subset of K_1 . Let ρ_1 be the least congruence on F_1 containing τ_1 . In a chain of form (1) for $u, v \in F_1$, if $u_i = a$ and $v_i = x_{ab}bu_{ab}$ or $u_i = b$ and $v_i = y_{ab}av_{ab}$ for $a, b \in S$ and $x_{ab}, y_{ab}, u_{ab}, v_{ab} \in Z_1$, call step i of (1) and x_{ab} or y_{ab} respectively an *insertion*. The reverse steps are deletions. Index the elements of $F(Z_1)$ in the chain (1) as in §2. Suppose $(x_{ab})_m$ is a central insertion-deletion; the transition involved in the insertion step being $a \rightarrow (x_{ab})_m b(u_{ab})_r$. It can be checked that there is a chain of form (1), obtainable from (1), with $(x_{ab})_m$ as a central insertion-deletion and with no step between the insertion and deletion of $(x_{ab})_m$ that involves the transition $b(u_{ab})_r \rightarrow y_{ab}a$. A similar statement applies for the other possible central insertion-deletions. With this observation, Lemma 2.1 can be readily generalised. The generalisation of the remainder of the construction is straightforward.

Note that the lack of symmetry in the existential conditions used to define bisimple semigroups is not essential. However with symmetrical conditions the proof of generalised Lemma 2.1 appears more cumbersome.

The construction for bisimple regular semigroups is a straightforward generalisation of the above.

THEOREM 3.1. *Let C be the category of simple, bisimple or bisimple regular semigroups. Then there are no nontrivial $C(X)$, $C(S)$ or $C(\chi)$ where X is a set, S is a semigroup and χ is a family of objects in C .*

It is known [2] that free products of any families exist in the category C of orthodox semigroups. It can therefore be readily checked that there exist free orthodox semigroups on some semigroups; for suppose χ is a family of orthodox semigroups and G is a free product of χ in the category of semigroups, then $C(G) = C(\chi)$. However free orthodox semigroups do not exist on any nonempty set [7].

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vols. 1 and 2, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R. I., 1961, 1967. MR 36 #1558.
2. I. Imaoka, *Free products of orthodox semigroups*, Mem. Fac. Lit. Sci. Shimane Univ. Natur. Sci. 10 (1976), 1–6.
3. D. B. McAlister, *A homomorphism theorem for semigroups*, J. London Math. Soc. 43 (1968), 355–366. MR 37 #329.
4. B. Mitchell, *Theory of categories*, Academic Press, New York, 1965. MR 34 #2647.
5. G. B. Preston, *Inverse semigroups: some open questions*, Proceedings of a Symposium on Inverse Semigroups and their Generalisations (Northern Illinois Univ., DeKalb, Ill., 1973), Northern Illinois Univ., DeKalb, Ill., 1973, pp. 122–139. MR 54 #2848.
6. H. E. Scheiblich, *Free inverse semigroups*, Proc. Amer. Math. Soc. 38 (1973), 1–7. MR 46 #9196.
7. P. G. Trotter, *Free objects in some categories of semigroups*, Conf. on the Algebraic Theory of Semigroups in Honour of A. H. Clifford (Tulane University, New Orleans, La., 1978).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TASMANIA, HOBART, TASMANIA, AUSTRALIA