A PROJECTIVE CHARACTERIZATION FOR SKT-MODULES

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ABSTRACT. In this paper a class of abelian groups (SKT-modules), which includes the torsion totally projective groups, S -groups, and balanced projectives is shown to be a subclass of a projective class of groups with respect to a naturally defined class of short exact sequences called the ch-projective modules and ch-pure sequences, respectively. Every Z_p -module has a ch-pure projective resolution and every reduced ch-projective module is a summand of a SKT-module. It is finally shown that a Z_p -module M is ch-projective if and only if, for every ordinal α , the two Z_p -modules $p^\alpha M$ and $M/p^\alpha M$ are both ch-projective.

1. Definitions. If p is a prime number then the ring of all rational numbers a/b with b relatively prime to p will be denoted by Z_p .

The category of Z_p -modules are those abelian groups with the property that multiplication by a prime other than the prime p is an automorphism of the group. If M is a Z_p -module then (i) the torsion submodule of M is the maximal torsion subgroup of M ; (ii) M is torsion-free if and only if the torsion submodule of M is (0); (iii) M is reduced if and only if M is a reduced group; and (iv) M is divisible if and only if M is a divisible group.

Z and Q will denote the groups of integers and rational numbers respectively. The group $\text{Ext}(Q/Z_p, *)$ will be denoted by $c(*)$.

The limit ordinal λ is a limit ordinal cofinal with ω if there is a sequence of smaller ordinals β_i such that $\lambda = \sup \beta_i$. Otherwise λ is said to be not cofinal with ω .

The family of vector spaces $K(p^\lambda, G)$ is defined for every group G to be $p^\lambda c(G/p^\lambda G)/p^{\lambda+1}c(G/p^\lambda G)$ where λ is any ordinal and p is a prime.

Warfield [1] and [2] introduced two classes of groups, balanced projectives and S -groups, which will play an important role in this paper. For the limit ordinal λ , a Z_p -module M is a λ -elementary balanced projective if and only if $p^\lambda M \cong Z_p$ and $M/p^\lambda M$ is a totally projective p -group. The torsion subgroup of M is called a λ -elementary S -group. A Z_p -module M is a balanced projective if and only if it is isomorphic to the direct sum of a totally projective p -group and λ -elementary balanced projectives for various limit ordinals λ . The torsion subgroup of a balanced projective is called an S -group.

A module M is an SKT-module if and only if there is a balanced projective K and an S -group S such that M is isomorphic to $K \oplus S$.

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2. ch-projectives. Since the objective is to find a projective characterization for the SKT-modules; for the present, a short exact sequence $e: 0 \rightarrow A \rightarrow B \xrightarrow{\pi} C \rightarrow 0$ is said to be an SKT-pure sequence if for every SKT-module M , the natural map $\pi^*: \text{Hom}(M, B) \rightarrow \text{Hom}(M, C)$ is surjective. It will be shown that a sequence $e: 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is SKT-pure if and only if the sequences $0 \rightarrow p^\alpha A \rightarrow p^\alpha B \rightarrow p^\alpha C \rightarrow 0$ and $0 \rightarrow p^\alpha c(A) \rightarrow p^\alpha c(B) \rightarrow p^\alpha c(C) \rightarrow 0$ are exact for every ordinal α .

THEOREM 2.1. *If the sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is SKT-pure then for every ordinal α , the sequences $0 \rightarrow p^\alpha A \rightarrow p^\alpha B \rightarrow p^\alpha C \rightarrow 0$ and $0 \rightarrow p^\alpha c(A) \rightarrow p^\alpha c(B) \rightarrow p^\alpha c(C) \rightarrow 0$ are exact.*

PROOF. Suppose that the sequence $e: 0 \rightarrow A \rightarrow B \xrightarrow{\pi} C \rightarrow 0$ is SKT-pure. It will first be shown that the natural maps $\pi: p^\alpha B \rightarrow p^\alpha C$, $\pi: p^\alpha B[p] \rightarrow p^\alpha C[p]$, and $\pi: p^\alpha c(B) \rightarrow p^\alpha c(C)$ are all surjective for each ordinal α where for a group G , $G[p]$ is the set of elements with order p .

Let α be an ordinal and $\alpha = \lambda + n$ where λ is a limit ordinal and $0 \leq n < \omega$. Let M be a λ -elementary balanced projective and m a generator of the free Z_p -module $p^\alpha M$. If $c \in p^\alpha C$ then there is a homomorphism $f: p^\alpha M \rightarrow p^\alpha C$ such that $f(m) = c$. Since f does not decrease heights, and $p^\alpha M$ is a p -nice submodule of M , and $M/p^\alpha M$ is a totally projective p -group; there is an extension $f^*: M \rightarrow C$ of f [2, 1.13]. Since the sequence e is SKT-pure, there is a homomorphism $g: M \rightarrow B$ such that $\pi g = f^*$; furthermore, $g(m) \in p^\alpha B$ and $\pi(g(m)) = c$. Thus $\pi: p^\alpha B \rightarrow p^\alpha C$ is surjective.

Similarly if $c \in p^\alpha c(C)$ then there is a homomorphism $f: M \rightarrow c(C)$ such that $f(m) = c$. Let S be the torsion submodule of M from which it follows that $f(S) \subseteq C$. Thus the restriction of f to S is a homomorphism from the S -group S to C . There is a homomorphism $g: S \rightarrow B$ such that $\pi g = f: S \rightarrow C$. Since M/S is torsion-free and divisible, g lifts to a homomorphism $g^*: M \rightarrow c(B)$; furthermore, $\pi g^* = f: M \rightarrow c(C)$, [1]. Since $g^*(m) \in p^\alpha c(B)$ and $\pi(g(m)) = c$, the map $\pi: p^\alpha c(B) \rightarrow p^\alpha c(C)$ is surjective.

Let $c \in p^\alpha C[p]$ and $H = M/p^{\alpha+1}M$. H is a totally projective p -group such that $p^\alpha H$ is cyclic of order p . If n is the generator of $p^\alpha H$ then there is a homomorphism $f: H \rightarrow C$ such that $f(n) = c$. There exists a homomorphism $g: H \rightarrow B$ such that $\pi g = f: H \rightarrow C$; furthermore, $g(n) \in p^\alpha B[p]$ and $\pi(g(n)) = c$. Thus the map $\pi: p^\alpha B[p] \rightarrow p^\alpha C[p]$ is surjective.

The desired result will follow from [2, 2.1].

DEFINITION. A sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is balanced if and only if the induced sequence $0 \rightarrow p^\alpha A \rightarrow p^\alpha B \rightarrow p^\alpha C \rightarrow 0$ is exact for every ordinal α .

DEFINITION. A sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is ch-pure if and only if the sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and $0 \rightarrow c(A) \rightarrow c(B) \rightarrow c(C) \rightarrow 0$ are balanced.

DEFINITION. A module M is ch-projective if and only if for every ch-pure sequence $0 \rightarrow A \rightarrow B \xrightarrow{\pi} C \rightarrow 0$, the natural map $\pi^*: \text{Hom}(M, B) \rightarrow \text{Hom}(M, C)$ is surjective.

THEOREM 2.2. *Divisible modules and SKT-modules are ch-projective.*

PROOF. If a sequence is ch-pure then it is balanced and by [1], divisible modules and balanced projectives are ch-projective.

To show that S -groups are ch-projective, only a λ -elementary S -group, S , needs to be considered. If S is the torsion subgroup of the λ -elementary balanced projective M then M/S is torsion-free and divisible. Since the sequence $0 \rightarrow p^\infty A \rightarrow p^\infty B \rightarrow p^\infty C \rightarrow 0$ is split exact, we may assume that A , B and C are reduced modules and submodules of their cotorsion envelopes $c(A)$, $c(B)$ and $c(C)$ respectively. The fact that module C is a submodule of $c(C)$ means that we can consider any homomorphism of S into C as a homomorphism from S into $c(C)$; furthermore, M/S being torsion-free and $c(C)$ being cotorsion implies that any homomorphism from S into $c(C)$ can be lifted to a homomorphism from M into $c(C)$. If $f: S \rightarrow C$ is a homomorphism then there is a homomorphism $f': M \rightarrow c(C)$ such that f' and f agree on S . Since M is a balanced projective module and the sequence $0 \rightarrow c(A) \rightarrow c(B) \rightarrow c(C) \rightarrow 0$ is balanced, there is a homomorphism $g: M \rightarrow c(B)$ such that $\pi g = f'$. The map g must send the torsion subgroup S into the torsion subgroup of $c(B)$. Thus $g(S) \subseteq B$ and the restriction of g to S is a homomorphism from S into B such that $\pi g = f: S \rightarrow C$.

Theorems 2.1 and 2.3 imply that the class of SKT-pure sequences coincides with the class of ch-pure sequences.

One might conjecture that if either $e_1: 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ or $e_2: 0 \rightarrow c(A) \rightarrow c(B) \rightarrow c(C) \rightarrow 0$ is balanced then the other shall be balanced. However the following two examples show that both e_1 and e_2 must be balanced in order to show that e_1 is ch-pure. To show that e_1 being balanced does not imply that e_2 is balanced, take C to be any S -group and e_1 a balanced projective resolution of C . By the conjecture, e_1 would be a ch-pure sequence; consequently it would be split exact; hence every S -group would be a totally projective group, which is not true. To show that e_2 being balanced does not imply that e_1 is balanced, take A to be any reduced group, $B = c(A)$ and C will be torsion-free and divisible. The sequence e_2 reduces to $0 \rightarrow c(A) \rightarrow c(B) \rightarrow 0$ which is balanced. The conjecture would imply that the sequence e_1 would be ch-pure and split. Thus $c(A)$ would not be reduced.

COROLLARY 2.3. *The sequence $0 \rightarrow A \rightarrow B \xrightarrow{\pi} C \rightarrow 0$ is ch-pure if and only if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is balanced and the map $\pi: p^\lambda c(B) \rightarrow p^\lambda c(C)$ is surjective for every limit ordinal λ such that λ is not cofinal with ω .*

PROOF. Suppose that the sequence $e: 0 \rightarrow A \rightarrow B \xrightarrow{\pi} C \rightarrow 0$ is balanced. We may assume that A , B and C are reduced and submodules of $c(A)$, $c(B)$ and $c(C)$ respectively. To show that the sequence e is ch-pure it will be sufficient to show that for every ordinal α the map $\pi: p^\alpha c(B) \rightarrow p^\alpha c(C)$ is surjective. Let $\alpha = \lambda + n$ where λ is a limit ordinal and $0 \leq n < \omega$.

Case I. If λ is not cofinal with ω then the map $\pi: p^\lambda c(B) \rightarrow p^\lambda c(C)$ being surjective implies that the map $\pi: p^\alpha c(B) \rightarrow p^\alpha c(C)$ is surjective.

Case II. Suppose λ is cofinal with ω . Let M be a λ -elementary balanced projective and m a generator of the free Z_p -module $p^\alpha M$. If $c \in p^\alpha c(C)$ then there

is a homomorphism $f: M \rightarrow c(C)$ such that $f(m) = c$. Let S be the torsion submodule of M and let f' be the restriction of f to S . Thus f' is a homomorphism from S to C . Since λ is cofinal with ω , S is a totally projective p -group and there is a homomorphism $g: S \rightarrow B$ such that $\pi g = f'$. The homomorphism g lifts to a homomorphism $g': M \rightarrow c(B)$ since $c(B)$ is cotorsion and M/S is torsion-free. Since M/S is torsion-free and divisible, $\pi g' = f: M \rightarrow c(C)$. Thus $g'(m) \in p^\omega c(B)$ and $\pi(g'(m)) = c$. This implies that $\pi: p^\omega c(B) \rightarrow p^\omega c(C)$ is surjective.

Another interesting characterization of ch-purity is the following:

THEOREM 2.4. *A sequence $0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0$ is ch-pure if and only if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is balanced and the sequence $0 \rightarrow K(p^\lambda, A) \xrightarrow{i} K(p^\lambda, B) \xrightarrow{\pi} K(p^\lambda, C) \rightarrow 0$ is exact for every limit ordinal λ such that λ is not cofinal with ω .*

PROOF. Suppose that the sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is ch-pure. Then the following commutative diagram has exact rows and columns. (λ is any ordinal).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & c(p^\lambda A) & \rightarrow & c(p^\lambda B) & \rightarrow & c(p^\lambda C) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & p^\lambda c(A) & \rightarrow & p^\lambda c(B) & \rightarrow & p^\lambda c(C) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 e^*: 0 & \rightarrow & p^\lambda c(A/p^\lambda A) & \rightarrow & p^\lambda c(B/p^\lambda B) & \rightarrow & p^\lambda c(C/p^\lambda C) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
 \end{array}$$

Since $p^\lambda c(C/p^\lambda C)$ is torsion-free, the sequence e^* is pure exact. This implies that the sequence

$$0 \rightarrow K(p^\lambda, A) \rightarrow K(p^\lambda, B) \rightarrow K(p^\lambda, C) \rightarrow 0$$

is exact.

To prove the converse, it will be sufficient to show that $\pi: p^\lambda c(B) \rightarrow p^\lambda c(C)$ is surjective for every limit ordinal λ such that λ is not cofinal with ω . Let X_0 be a basis for $K(p^\lambda, A)$. Since the sequence $0 \rightarrow K(p^\lambda, A) \xrightarrow{i} K(p^\lambda, B) \xrightarrow{\pi} K(p^\lambda, C) \rightarrow 0$ is exact, the image of X_0 , $\iota(X_0)$, in $K(p^\lambda, B)$ can be extended by a set X such that $\iota(X_0) \cup X$ is a basis for $K(p^\lambda, B)$ and $\pi(X)$ is a basis for $K(p^\lambda, C)$.

Let $\underline{X}_0$ and $\underline{X}$ be subsets of $p^\lambda c(B)$ which are mapped bijectively by the natural map $\sigma_\lambda: p^\lambda c(B) \rightarrow p^\lambda c(B/p^\lambda B)/p^{\lambda+1}c(B/p^\lambda B) = K(p^\lambda, B)$ onto $\iota(X_0)$ and X respectively. There is a torsion-free submodule L of $p^\lambda c(B)$ such that $p^\lambda c(B) = c(p^\lambda B) \oplus L$ where L is the p -adic completion of the free module generated by $\underline{X}_0 \cup \underline{X}$. Similarly there is a submodule L' of $p^\lambda c(C)$ such that $p^\lambda c(C) = c(p^\lambda C) \oplus L'$ and L' is the p -adic completion of the free module generated by $\pi(\underline{X})$. Clearly there is a surjective homomorphism $\gamma: L \rightarrow L'$ such that for each $x \in \underline{X}$, $\gamma(x) = \pi(x)$. Since the sequence $0 \rightarrow p^\lambda A \rightarrow p^\lambda B \rightarrow p^\lambda C \rightarrow 0$ is exact, $\pi: c(p^\lambda B) \rightarrow c(p^\lambda C)$ is surjective. Thus a homomorphism $\gamma': p^\lambda c(B) \rightarrow p^\lambda c(C)$ can be constructed such that γ' is surjective and γ' agrees with π on the submodule $c(p^\lambda B) + [\underline{X}_0 \cup \underline{X}]$. Since

$p^\lambda c(B)/(c(p^\lambda B) + [X_0 \cup X])$ is torsion-free and divisible, $\pi = \gamma'$ and $\pi: p^\lambda c(B) \rightarrow p^\lambda c(C)$ is surjective.

THEOREM 2.5. *If N is a Z_p -module then there is a ch-pure sequence $e: 0 \rightarrow A \rightarrow B \rightarrow N \rightarrow 0$ such that B is the direct sum of a divisible module and an SKT-module; i.e., B is a ch-projective module. If N is reduced then B can be chosen to be an SKT-module.*

PROOF. Let $N = D \oplus R$ where D is the maximal divisible submodule of N and R is a reduced module. There are balanced sequences $0 \rightarrow A_1 \rightarrow P_1 \rightarrow R \xrightarrow{\pi_1} 0$ and $0 \rightarrow A_2 \rightarrow P_2 \xrightarrow{\pi_2} c(R) \rightarrow 0$ such that P_1 and P_2 are balanced projectives. Let S be the torsion subgroup of P_2 , $B = D \oplus P_1 \oplus S$ and define $\pi: B \rightarrow N$ by $\pi(d, p, s) = d + \pi_1(p) + \pi_2(s)$. If A is the kernel of π then clearly the sequence $0 \rightarrow A \rightarrow B \xrightarrow{\pi} N \rightarrow 0$ is balanced; thus we need only demonstrate that for the ordinal α , the map $\pi: p^\alpha c(B) \rightarrow p^\alpha c(N)$ is surjective. In fact, it may be assumed that α is an infinite ordinal. The injection map $\iota: S \rightarrow P_2$ induces the isomorphism $\iota: p^\alpha c(S) \rightarrow p^\alpha c(P_2)$. Consequently, the map $\pi_2: p^\alpha c(S) \rightarrow p^\alpha c(R) \cong p^\alpha c(N)$ is surjective. This implies that the map $\pi: p^\alpha c(B) \rightarrow p^\alpha c(N)$ is surjective.

THEOREM 2.6. *If M is a Z_p -module and α an ordinal then M is ch-projective if and only if $p^\alpha M$ and $M/p^\alpha M$ are ch-projective.*

PROOF. Clearly if M is ch-projective then $p^\alpha M$ and $M/p^\alpha M$ are ch-projective.

Suppose $p^\alpha M$ and $M/p^\alpha M$ are ch-projective and $e: 0 \rightarrow A \rightarrow B \xrightarrow{\pi} C \rightarrow 0$ is a ch-pure sequence. First it will be shown that the sequence $e_\alpha: 0 \rightarrow p^\alpha A \rightarrow p^\alpha B \xrightarrow{\pi} p^\alpha C \rightarrow 0$ is ch-pure. Clearly the sequence e_α is balanced. Thus to show that e_α is ch-pure it is required only to show that the map $\pi: p^\lambda c(p^\alpha B) \rightarrow p^\lambda c(p^\alpha C)$ is surjective for all limit ordinals $\lambda > 0$ such that λ is not cofinal with ω . There are torsion-free modules X and Y such that $p^\alpha c(B) = c(p^\alpha B) \oplus X$ and $p^\alpha c(C) = c(p^\alpha C) \oplus Y$. Since λ is an infinite ordinal, $p^\lambda p^\alpha c(B) = p^\lambda c(p^\alpha B)$ and $p^\lambda p^\alpha c(C) = p^\lambda c(p^\alpha C)$. It follows that the map $\pi: p^\lambda c(p^\alpha B) \rightarrow p^\lambda c(p^\alpha C)$ is surjective because the map $\pi: p^\lambda p^\alpha c(B) \rightarrow p^\lambda p^\alpha c(C)$ is surjective.

Let $f \in \text{Hom}(M, C)$. Then $f(p^\alpha M) \subseteq p^\alpha C$. There exists a homomorphism $g: p^\alpha M \rightarrow p^\alpha B$ because $p^\alpha M$ is ch-projective and the sequence $0 \rightarrow p^\alpha A \rightarrow p^\alpha B \rightarrow p^\alpha C \rightarrow 0$ is ch-pure. The homomorphism g lifts to a homomorphism $g^*: M \rightarrow B$. Let $h = f - \pi g^*: M \rightarrow C$ and note that $h(p^\alpha M) = 0$. Consequently, we can consider h as a homomorphism from $M/p^\alpha M$ to C . Since $M/p^\alpha M$ is ch-projective, there is a homomorphism $\gamma: M \rightarrow M/p^\alpha M \rightarrow B$ such that $\pi\gamma = h = f - \pi g^*$ or, equivalently, $f = \pi(\gamma + g^*)$. Thus M is ch-projective.

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