DIVIDED DIFFERENCES AND SYSTEMS OF NONHARMONIC FOURIER SERIES¹

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ABSTRACT. Suppose that $\omega_{n,0}, \omega_{n,1}, \dots, \omega_{n,k}$ are distinct complex numbers with $|n - \omega_{n,j}| < \delta$ for all $n \in \mathbb{Z}$, $j = 0, 1, \dots, k$. We show that if $\delta > 0$ is small enough then, given complex numbers $c_{n,j}$ ($n \in \mathbb{Z}, j = 0, 1, \dots, k$) there exists $f \in L^2(-(k+1)\pi, (k+1)\pi)$ with

$$\int_{-(k+1)\pi}^{(k+1)\pi} f(t) e^{-i\omega_{n,j}t} dt = c_{n,j} \quad \text{for } n \in \mathbb{Z}, j = 0, 1, \dots, k$$

if and only if certain "divided differences" involving the $c_{n,j}$'s and the $\omega_{n,j}$'s are square summable. This extends a classical theorem of Paley and Wiener, which is equivalent to the case $k = 0$ above.

0. Introduction. A classical theorem of Paley and Wiener [5, p. 113] states that if $\delta < 1/\pi^2$ and $\{\omega_n\}$ is a sequence of reals satisfying $|n - \omega_n| < \delta$ for $n \in \mathbb{Z}$, then the functions $e^{i\omega_n t}$ form a "Riesz basis" for $L^2[-\pi, \pi]$. This conclusion is equivalent to the following statement: Suppose $\{c_n\}$ ($n \in \mathbb{Z}$) is a sequence of complex numbers. Then the interpolation problem

$$\int_{-\pi}^{\pi} f(t) e^{-i\omega_n t} dt = c_n \quad (n \in \mathbb{Z}) \tag{0}$$

has a solution $f \in L^2[-\pi, \pi]$ if and only if $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$. (Kadec [3] has shown that this holds under the weaker hypothesis $\delta < \frac{1}{4}$, which is known to be the best possible result; see [4, p. 48].)

Young [8] has recently established an interesting generalization of this result; the purpose of the present paper is to give another generalization, of an entirely different sort. Consider the following problem: Suppose K is a positive integer, and the $\omega_{n,j}$ ($n \in \mathbb{Z}, 0 \leq j \leq K$) are complex numbers with each $\omega_{n,j}$ "close enough" to n . For what choices of complex numbers $c_{n,j}$ will the system

$$\int_{-(K+1)\pi}^{(K+1)\pi} f(t) e^{-i\omega_{n,j}t} dt = c_{n,j} \quad (n \in \mathbb{Z}, 0 \leq j \leq K) \tag{1}$$

have a solution $f \in L^2(-(K+1)\pi, (K+1)\pi)$? In Theorem 3 below we demonstrate the existence of $\delta = \delta_K > 0$ such that under the hypothesis $|n - \omega_{n,j}| < \delta_K$ ($n \in \mathbb{Z}, 0 \leq j \leq K$), the system (1) has a solution $f \in L^2(-(K+1)\pi, (K+1)\pi)$ if and only if certain "divided differences" involving the $c_{n,j}$ are square summable. (See Theorem 3 for the precise statement.)

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[*Note.* Throughout this paper “scalar” means “complex number”, “Hilbert space” means “complex Hilbert space”, etc.]

1. Riesz bases.

DEFINITIONS. Let $\{x_n\}$ and $\{y_n\}$ ($n \in \mathbf{Z}$) be sequences of elements of the Hilbert space H . Then $\{y_n\}$ is biorthogonal to $\{x_n\}$ if for $n, m \in \mathbf{Z}$

$$\langle x_n, y_m \rangle = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

The sequence $\{x_n\}$ is independent if there exists a sequence $\{y_n\}$ which is biorthogonal to $\{x_n\}$. The sequence $\{x_n\}$ is complete in H if 0 is the only element of H which is orthogonal to each x_n ($n \in \mathbf{Z}$). (Equivalently, $\{x_n\}$ is complete if and only if its linear span is dense in H .)

The following theorem is essentially a summary of various “well-known” facts, many of which are contained, at least implicitly, in [5, Chapter VII], and in [1]. It and the lemma which follows it are included here primarily for the convenience of the reader.

THEOREM 1. Let $\{x_n\}$ ($n \in \mathbf{Z}$) be a sequence of elements of the Hilbert space H . The following are equivalent:

(i) For scalars c_n ($n \in \mathbf{Z}$) the system $\langle y, x_n \rangle = c_n$ ($n \in \mathbf{Z}$) has a solution $y \in H$ if and only if $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$.

(ii) There exist $A, B > 0$ such that for any N and for any scalars $a_{-N}, \dots, a_N$,

$$A^2 \sum_{-N}^N |a_n|^2 \leq \left\| \sum_{-N}^N a_n x_n \right\|^2 \leq B^2 \sum_{-N}^N |a_n|^2.$$

Further, if $\{x_n\}$ is complete in H , then (i) and (ii) are each equivalent to each of the following:

(iii) There exists $\{y_n\}$ biorthogonal to $\{x_n\}$, with $\{y_n\}$ complete in H , and there exist $A, B > 0$ such that for any N and for any scalars $c_{-N}, \dots, c_N$

$$B^{-2} \sum_{-N}^N |c_n|^2 \leq \left\| \sum_{-N}^N c_n y_n \right\|^2 \leq A^{-2} \sum_{-N}^N |c_n|^2.$$

(iv) For any scalars a_n ($n \in \mathbf{Z}$) such that $\sum_{-\infty}^{\infty} |a_n|^2 < \infty$, $\sum_{-\infty}^{\infty} a_n x_n = \lim_{N \rightarrow \infty} \sum_{-N}^N a_n x_n$ exists in H , and for any $x \in H$ there exists a unique sequence $\{a_n\}$ with $\sum_{-\infty}^{\infty} |a_n|^2 < \infty$ and $x = \sum_{-\infty}^{\infty} a_n x_n$.

[*Note.* The limit in (iv) refers to the norm topology on H .]

PROOF. Suppose, to start with, that $\{x_n\}$ is complete in H . We show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Let $l^2 = l^2(\mathbf{Z})$ be the Hilbert space of square summable sequences, indexed by $\mathbf{Z}$; for any $\zeta \in l^2$ and $n \in \mathbf{Z}$, we will denote by ζ_n the n th component of ζ ; e_n will be the element of l^2 with $(e_n)_m = \delta_{n,m}$ ($n, m \in \mathbf{Z}$).

(i) $\Rightarrow$ (iii): Suppose (i). This says precisely that we can define a linear map $T: H \rightarrow l^2$ by

$$(Tx)_n = \langle x, x_n \rangle,$$

and that $T(H) = l^2$. The closed graph theorem implies that T is bounded, and the completeness of $\{x_n\}$ shows that T is one-to-one. Hence (by the open mapping theorem) T^{-1} exists and is bounded. Let $y_n = T^{-1}(e_n)$. Then

$$\langle y_n, x_m \rangle = (Ty_n)_m = (e_n)_m = \delta_{n,m},$$

so $\{y_n\}$ is biorthogonal to $\{x_n\}$. Since T is a Banach space isomorphism, the completeness of $\{e_n\}$ in l^2 gives completeness of $\{x_n\}$ in H . (See the comment after the definition of "complete".) Similarly, the two inequalities in (iii) are precisely the statement that T and T^{-1} are bounded.

(iii) $\Rightarrow$ (ii): Suppose (iii). Then we can define a (unique) bounded invertible linear map $T: H \rightarrow l^2$ by $Ty_n = e_n$. But now the Hilbert-space adjoint $T^*: l^2 \rightarrow H$ is also invertible, since $(T^*)^{-1} = (T^{-1})^*$. Since, by (iii),

$$\langle T^*e_n, y_m \rangle = \langle e_n, Ty_m \rangle = \langle e_n, e_m \rangle = \delta_{n,m} = \langle x_n, y_m \rangle,$$

the completeness of $\{y_n\}$ shows that $T^*e_n = x_n$. Hence (as above) boundedness of T^* and $(T^*)^{-1}$ gives (ii).

(ii) $\Rightarrow$ (iv): Suppose (ii). Then if $\sum_{-\infty}^{\infty} |a_n|^2 < \infty$, the limit $\lim_{N \rightarrow \infty} \sum_{-N}^N a_n x_n = \sum_{-\infty}^{\infty} a_n x_n$ certainly exists, by the completeness of H . So we may define a linear map $T: l^2 \rightarrow H$ by $Ta = \sum_{-\infty}^{\infty} a_n x_n$. Then T is bounded and bounded below, by (ii). Since $\{x_n\}$ is complete in H , $T(l^2)$ is dense in H ; since T is bounded below, $T(l^2)$ is closed. Thus $T(l^2) = H$; similarly T is one-to-one, being bounded below on a dense subspace of l^2 ; this gives (iv).

(iv) $\Rightarrow$ (i): Suppose (iv). For each positive integer N , define $T_N: l^2 \rightarrow H$ by $T_N a = \sum_{-N}^N a_n x_n$. Since

$$\|T_N a\| \leq \sum_{-N}^N |a_n| \|x_n\| \leq \left(\sum_{-N}^N \|x_n\|^2 \right)^{1/2} \|a\|,$$

each T_N is bounded. The hypothesis says that for each $a \in l^2$, $Ta = \lim_{N \rightarrow \infty} T_N a$ exists in H . Hence, by the Banach-Steinhaus theorem, $\{\|T_N\|\}$ is bounded, and thus T is bounded; the hypothesis also implies that T is one-to-one and onto. Hence, as above, T is invertible, so that T^* is also invertible; in particular $T^*(H) = l^2$. But $(T^*y)_n = \langle T^*y, e_n \rangle = \langle y, Te_n \rangle = \langle y, x_n \rangle$, so $T^*(H) = l^2$ gives (i).

Thus far we have proved Theorem 1 in the case that $\{x_n\}$ is complete in H . The equivalence of (i) and (ii) in general follows from this case:

Let H' be the smallest closed subspace of H containing each x_n ($n \in \mathbf{Z}$). Then $\{x_n\}$ is certainly complete in H' , so we have shown that (i) and (ii) are equivalent if H' is substituted for H . But the statement of (ii) is entirely unaffected by substituting H' for H , and (i) is not essentially affected: Let $P: H \rightarrow H'$ be the orthogonal projection with $PH = H'$; then for $y \in H$, $n \in \mathbf{Z}$ we have $\langle y, x_n \rangle = \langle Py, x_n \rangle$. Hence for a given sequence of scalars $\{c_n\}$, there exists $y \in H$ with $\langle y, x_n \rangle = c_n$ ($n \in \mathbf{Z}$) if and only if there exists $y \in H'$ with $\langle y, x_n \rangle = c_n$ ($n \in \mathbf{Z}$). Thus the equivalence of (i) and (ii) follows from their equivalence with H replaced by H' . Q.E.D.

DEFINITION. Let $\{x_n\}$ be a sequence of elements of the Hilbert space H . Then $\{x_n\}$ is a Riesz basis for H if

(i) $\sum_{-\infty}^{\infty} a_n x_n = \lim_{N \rightarrow \infty} \sum_{-N}^N a_n x_n$ exists in H for any sequence of scalars $\{a_n\}$ satisfying $\sum_{-\infty}^{\infty} |a_n|^2 < \infty$, and

(ii) for any $x \in H$ there exists a unique sequence of scalars $\{a_n\}$ with $\sum_{-\infty}^{\infty} |a_n|^2 < \infty$ and $x = \sum_{-\infty}^{\infty} a_n x_n$.

Thus a Riesz basis is in fact a (Schauder) basis for H ; Theorem 1 shows that if $\{x_n\}$ is a Riesz basis for H , then so is its biorthogonal set. (Note that in general a complete set of vectors has at most one biorthogonal set.)

The proof of the next lemma is essentially the same as that in [6, p. 208], although the statement is more general than that which occurs in [6]. (See also [5, p. 100].)

LEMMA 1. Suppose that $\{x_n\}$ is a Riesz basis for the Hilbert space H , and $A, B > 0$ are chosen so that for any N and for any scalars $a_{-N}, \dots, a_N$

$$A^2 \sum_{-N}^N |a_n|^2 \leq \left\| \sum_{-N}^N a_n x_n \right\|^2 \leq B^2 \sum_{-N}^N |a_n|^2.$$

(The existence of such A and B is guaranteed by Theorem 1.) Suppose further that $0 < A' < A$ and $\{y_n\}$ is a sequence of elements of H such that for any N and for any scalars $a_{-N}, \dots, a_N$

$$\left\| \sum_{-N}^N a_n y_n \right\|^2 \leq (A')^2 \sum_{-N}^N |a_n|^2.$$

Then $\{x_n + y_n\}$ is complete in H and for any N and for any scalars $a_{-N}, \dots, a_N$,

$$(A - A')^2 \sum_{-N}^N |a_n|^2 \leq \left\| \sum_{-N}^N a_n (x_n + y_n) \right\|^2 \leq (B + A')^2 \sum_{-N}^N |a_n|^2.$$

(In particular $\{x_n + y_n\}$ is also a Riesz basis for H .)

PROOF. The inequalities in the conclusion are clear:

$$\begin{aligned} (A - A') \left(\sum_{-N}^N |a_n|^2 \right)^{1/2} &\leq \left\| \sum_{-N}^N a_n x_n \right\| - \left\| \sum_{-N}^N a_n y_n \right\| \leq \left\| \sum_{-N}^N a_n (x_n + y_n) \right\| \\ &\leq \left\| \sum_{-N}^N a_n x_n \right\| + \left\| \sum_{-N}^N a_n y_n \right\| \leq (B + A') \left(\sum_{-N}^N |a_n|^2 \right)^{1/2}. \end{aligned}$$

Now define $T(\sum_{-N}^N a_n x_n) = \sum_{-N}^N a_n (x_n + y_n)$.

Since

$$\begin{aligned} \left\| \sum_{-N}^N a_n (x_n + y_n) \right\| &\leq \left\| \sum_{-N}^N a_n x_n \right\| + \left\| \sum_{-N}^N a_n y_n \right\| \\ &\leq \left\| \sum_{-N}^N a_n x_n \right\| + A' \left(\sum_{-N}^N |a_n|^2 \right)^{1/2} \\ &\leq \left(1 + \frac{A'}{A} \right) \left\| \sum_{-N}^N a_n x_n \right\|, \end{aligned}$$

we can extend T to a (unique, by the completeness of $\{x_n\}$) bounded linear $T: H \rightarrow H$. Let I denote the identity operator; then

$$\begin{aligned} \left\| (I - T) \sum_{-N}^N a_n x_n \right\| &= \left\| \sum_{-N}^N (-a_n y_n) \right\| \\ &\leq A' \left(\sum_{-N}^N |a_n|^2 \right)^{1/2} \leq \frac{A'}{A} \left\| \sum_{-N}^N a_n x_n \right\|. \end{aligned}$$

That is, $\|I - T\| \leq A'/A < 1$, so that T is invertible. I.e., T is a Banach space isomorphism, so the completeness of $\{x_n\}$ and the fact that $Tx_n = x_n + y_n$ show that $\{x_n + y_n\}$ is complete in H . Q.E.D.

2. Exponentials.

NOTATION. If f is a function in $L^2(\mathbf{R})$ with compact support and $z \in \mathbf{C}$ we define

$$\hat{f}(z) = (2\pi)^{-1/2} \int f(t) e^{-izt} dt.$$

For $r \geq 0$ let I_r denote the interval $-(r+1)\pi, (r+1)\pi$.

The following proposition is quite trivial; we insert it here simply because it would seem awkward to incorporate its proof into that of Theorem 2.

PROPOSITION. If $r \geq 0$, $f \in L^2(I_r)$ and n is a nonnegative integer, then there exists $g \in L^2(I_{r+n})$ with

$$\hat{g}(z) = (\sin \pi z)^n \hat{f}(z) \quad (z \in \mathbf{C}).$$

PROOF. We need only do the case $n = 1$; the general case then follows by iteration. But let $g(x) = (2i)^{-1}(f(x - \pi) - f(x + \pi))$; then clearly $f \in L^2(I_r)$ implies that $g \in L^2(I_{r+1})$, and one easily verifies that $\hat{g}(z) = \sin(z)\hat{f}(z)$. Q.E.D.

THEOREM 2. If K is a nonnegative integer then the functions

$$\{e^{int}, te^{int}, \dots, t^K e^{int}\} \quad (n \in \mathbf{Z})$$

form a Riesz basis for $L^2(I_K)$.

PROOF. First we show completeness. Theorem III in [4] states, in part, that if $\{\lambda_n\}$ ($n \in \mathbf{Z}$) is a sequence of distinct reals, $\Lambda(r)$ is the number of $n \in \mathbf{Z}$ with $|\lambda_n| \leq r$, and there exists a constant C so that

$$\int_1^R \frac{\Lambda(r)}{r} dr \geq 2R - \frac{1}{2} \log R + C \quad (r \geq 1)$$

then $\{e^{i\lambda_n t}\}$ is complete in $L^2[-\pi, \pi]$. This statement does not appear to be helpful in our situation. However, the proof of the theorem shows that if $f \in L^2[-\pi, \pi]$ and $\Lambda(r)$ is the number of real zeroes of $\hat{f}$ of modulus $\leq r$, and if Λ satisfies the above condition, then $f = 0$. The proof of this fact given in [4] remains valid if the zeroes of the entire function $\hat{f}$ are counted with multiplicities. Suppose $f \in L^2(I_K)$ and $\langle f, t^j e^{int} \rangle = 0$ for $n \in \mathbf{Z}$, $0 \leq j \leq K$; we need to show that $f = 0$. Let $g(t) = f((K+1)t)$, so that $g \in L^2(-\pi, \pi)$, and let $\Lambda(r)$ be the number of real zeroes of $\hat{g}$, of modulus $\leq r$. Now, $\hat{g}$ has a zero of order at least $K+1$ at each

point $(K + 1)n$ ($n \in \mathbf{Z}$). Hence, if $[x]$ denotes the greatest integer $\leq x$, then

$$\Lambda(r) \geq 2(K + 1)[r / (K + 1)] + K + 1.$$

It follows that for any positive integer n we have

$$\begin{aligned} \int_{n(K+1)}^{(n+1)(K+1)} \frac{\Lambda(r)}{r} dr &\geq \int_{n(K+1)}^{(n+1)(K+1)} \frac{(2n+1)(K+1)}{r} dr \\ &= (2n+1)(K+1) \int_0^{(K+1)/2} \left(\frac{1}{(n+\frac{1}{2})(K+1)+t} + \frac{1}{(n+\frac{1}{2})(K+1)-t} \right) dt \\ &= (2n+1)^2(K+1)^2 \int_0^{(K+1)/2} \frac{dt}{(n+\frac{1}{2})^2(K+1)^2 - t^2} \\ &\geq (2n+1)^2(K+1)^2 \int_0^{(K+1)/2} \frac{dt}{(n+\frac{1}{2})^2(K+1)^2} = 2(K+1). \end{aligned}$$

This shows that for some constant C

$$\int_1^R \frac{\Lambda(r)}{r} dr \geq 2R + C \quad (R \geq 1).$$

Thus the result of Levinson quoted above shows that $g = 0$, and hence $f = 0$; this finishes the proof of the completeness of the set $\{t^j e^{int}\}$ ($n \in \mathbf{Z}$, $0 \leq j \leq K$) in $L^2(I_K)$.

Now, by Theorem 1, to show that this set is a Riesz basis we need only show that for any choice of scalars c_{nj} ($n \in \mathbf{Z}$, $0 \leq j \leq K$) the system

$$\langle f, t^j e^{int} \rangle = c_{nj} \quad (n \in \mathbf{Z}, 0 \leq j \leq K)$$

has a solution $f \in L^2(I_K)$ if and only if $\sum_{nj} |c_{nj}|^2 < \infty$.

Since $\langle f, t^j e^{int} \rangle = \langle t^j f, e^{int} \rangle$ and $t^j f \in L^2(I_K)$ if $f \in L^2(I_K)$, the necessity of the condition $\sum_{nj} |c_{nj}|^2 < \infty$ is evident. We show sufficiency by induction on K ; the case $K = 0$ is also perfectly standard.

For the induction step, suppose we have proved the theorem for $K = K_0$, and suppose that $\{c_{nj}\}$ ($n \in \mathbf{Z}$, $0 \leq j \leq K_0 + 1$) is a sequence of scalars with $\sum_{nj} |c_{nj}|^2 < \infty$; we need to find $f \in L^2(I_{K_0+1})$ with

$$\int_{I_{K_0+1}} f(t) t^j e^{-int} dt = c_{nj} \quad (n \in \mathbf{Z}, 0 \leq j \leq K_0 + 1). \quad (1)$$

By the induction hypothesis we can find $f_1 \in L^2(I_{K_0})$ with

$$\int_{I_{K_0}} f(t) t^j e^{-int} dt = c_{nj} \quad (n \in \mathbf{Z}, 0 \leq j \leq K_0). \quad (2)$$

Extend f_1 to be zero on $I_{K_0+1} - I_{K_0}$, so $f_1 \in L^2(I_{K_0+1})$. Now since $t^{K_0+1} f_1(t) \in L^2(I_{K_0+1})$ and the c_{nj} are square summable we can find $g \in L^2(I_0) = L^2(-\pi, \pi)$ such that

$$\begin{aligned} \int_{-\pi}^{\pi} g(t) e^{-int} dt &= ((i\pi)^{(K_0+1)} (K_0 + 1)!)^{-1} \left(c_{n, K_0+1} - \int_{I_{K_0+1}} f_1(t) t^{K_0+1} e^{-int} dt \right) \\ &\quad (n \in \mathbf{Z}). \quad (3) \end{aligned}$$

By the proposition above we can find $f_2 \in L^2(I_{K_0+1})$ with $\hat{f}_2(z) = (\sin \pi z)^{K_0+1} \hat{g}(z)$ ($z \in \mathbb{C}$). Let $F = \hat{f}_2$; the derivatives of the entire function F are given by

$$F^{(j)}(z) = (2\pi)^{-1/2} (-i)^j \int_{I_{K_0+1}} f_2(t) t^j e^{-izt} dt. \quad (4)$$

But since $F(z) = (\sin \pi z)^{K_0+1} \hat{g}(z)$ we see that for $n \in \mathbb{Z}$

$$F^{(j)}(n) = 0 \quad \text{if } 0 \leq j \leq K_0, \quad F^{(K_0+1)}(n) = \pi^{(K_0+1)} (K_0 + 1)! \hat{g}(n).$$

Combining (3) and (4) yields

$$\int_{I_{K_0+1}} f_2(t) t^j e^{-int} dt = \begin{cases} 0 & (n \in \mathbb{Z}, 0 \leq j \leq K_0), \\ c_{n, K_0+1} - \int_{I_{K_0+1}} f_1(t) t^{K_0+1} e^{-int} dt & (n \in \mathbb{Z}, j = K_0 + 1). \end{cases}$$

Hence (2) shows that if $f = f_1 + f_2$, then f is a solution to (1). Q.E.D.

LEMMA 2. Let $\varphi_n \in C^2(I_{K+1})$ ($n \in \mathbb{Z}$), let $\varepsilon > 0$, and suppose that $\|\varphi_n\|_\infty \leq \varepsilon$, $\|\varphi'_n\|_\infty \leq \varepsilon$, and $\|\varphi''_n\|_\infty < \varepsilon$ for each $n \in \mathbb{Z}$. Then for any scalars $\{a_n\}$ and any N

$$\int_{I_K} \left| \sum_{-N}^N a_n e^{int} \varphi_n(t) \right|^2 dt \leq \varepsilon^2 M^2 \sum_{-N}^N |a_n|^2,$$

where M depends only on K .

PROOF. Fix $\psi \in C^2(\mathbb{R})$ with support $(\psi) \subseteq I_{K+1}$, $0 \leq \psi \leq 1$, and $\psi(t) = 1$ for $t \in I_K$. Let $f_n(t) = e^{int} \varphi_n(t) \psi(t)$; then

$$\int_{I_{K+1}} |f_n(t)|^2 dt \leq 2(K+2)\pi\varepsilon^2$$

For $n \neq m$, let $F_{n,m} = \varphi_n \bar{\varphi}_m \psi^2$. Then

$$F_{n,m}(\pm(K+1)\pi) = F'_{n,m}(\pm(K+1)\pi) = 0,$$

so integrating twice by parts gives

$$\begin{aligned} \left| \int_{I_{K+1}} f_n \bar{f}_m dt \right| &= \left| \int_{I_{K+1}} F_{n,m}(t) e^{i(n-m)t} dt \right| \\ &= \left| \int_{I_{K+1}} \frac{e^{i(n-m)t}}{(n-m)^2} F''_{n,m}(t) dt \right| \\ &\leq \frac{1}{(n-m)^2} \int_{I_{K+1}} |F''_{n,m}(t)| dt. \end{aligned}$$

By Leibniz' formula and the hypotheses on $\{\varphi_n\}$ we can certainly find M' (depending only on K) with

$$\int_{I_{K+1}} |F''_{n,m}(t)| dt < M' \varepsilon^2 \quad (n, m \in \mathbb{Z}, n \neq m).$$

But now

$$\begin{aligned}
 \int_{I_K} \left| \sum_{-N}^N a_n e^{int} \varphi_n(t) \right|^2 dt &\leq \int_{I_{K+1}} \left| \sum_{-N}^N a_n f_n \right|^2 dt \\
 &= \int_{I_{K+1}} \left[\sum_{-N}^N |a_n|^2 |f_n|^2 + \sum_{\substack{n,m=-N \\ n \neq m}}^N a_n \bar{a}_m f_n \bar{f}_m \right] dt \\
 &\leq \sum_{-N}^N |a_n|^2 \int_{I_{K+1}} |f_n|^2 dt + \sum_{\substack{n,m=-N \\ n \neq m}}^N |a_n a_m| \left| \int_{I_{K+1}} f_n \bar{f}_m dt \right| \\
 &\leq \varepsilon^2 2(K+2)\pi \sum_{-N}^N |a_n|^2 + \varepsilon^2 M' \sum_{\substack{n,m=-N \\ n \neq m}}^N \frac{|a_n a_m|}{(n-m)^2} \\
 &\leq \varepsilon^2 2(K+2)\pi \sum_{-N}^N |a_n|^2 + \varepsilon^2 M' \sum_{m \neq 0} \left(\frac{1}{m^2} \sum_n |a_{n+m} a_n| \right) \\
 &\leq \varepsilon^2 2(K+2)\pi \sum_{-N}^N |a_n|^2 + \varepsilon^2 M' \sum_{m \neq 0} \left(\frac{1}{m^2} \sum_{n=-N}^N |a_n|^2 \right) \\
 &= \varepsilon^2 \left(2(K+2)\pi + \frac{M' \pi^2}{3} \right) \sum_{n=-N}^N |a_n|^2. \quad \text{Q.E.D.}
 \end{aligned}$$

We now introduce notation for divided differences. If $f(z)$ is a function of a complex variable and $z_0, \dots, z_j$ are distinct complex numbers, let $f[z_0] = f(z_0)$ and

$$f[z_0, \dots, z_j] = \frac{f[z_0, \dots, z_{j-1}] - f[z_1, \dots, z_j]}{z_0 - z_j}.$$

In order to fit exponentials into this scheme, we write $e^{itz} = \exp_t(z)$.

In the theorem below $\{\omega_{n,j}\}$ will be a fixed set of complex numbers ($n \in \mathbf{Z}$, $0 \leq j \leq K$), such that $\omega_{n,j} \neq \omega_{n,k}$ if $j \neq k$. If $\{c_{n,j}\}$ is any set of complex numbers ($n \in \mathbf{Z}$, $0 \leq j \leq K$), write $c_j^n = c_{n,j}$, and

$$c_{j, \dots, k}^n = \frac{c_{j, \dots, k-1}^n - c_{j+1, \dots, k}^n}{\omega_{n,j} - \omega_{n,k}} \quad (n \in \mathbf{Z}, 0 \leq j < k \leq K).$$

We can now state

THEOREM 3. *To each nonnegative integer K there corresponds $\delta_K > 0$ such that if $\{\omega_{n,j}\}$ ($n \in \mathbf{Z}$, $0 \leq j \leq K$) is a set of distinct complex numbers with $|n - \omega_{n,j}| < \delta_K$ for $n \in \mathbf{Z}$ and $0 \leq j \leq K$, then the following conclusion holds:*

The system

$$\int_{I_K} f(t) e^{-it\omega_{n,j}} dt = c_{n,j} \quad (n \in \mathbf{Z}, 0 \leq j \leq K) \quad (5)$$

has a solution $f \in L^2(I_K)$ if and only if

$$\sum_{n=-\infty}^{\infty} (|c_0^n|^2 + |c_{0,1}^n|^2 + \dots + |c_{0,1,\dots,K}^n|^2) < \infty; \quad (6)$$

further, if (5) has a solution it is unique.

PROOF. Fix K . Clearly, given $f \in L^2(I_K)$ and scalars $\{c_{n,j}\}$, the statement (5) holds if and only if

$$\int_{I_K} f(t) \exp_t[\omega_{n,0}, \dots, \omega_{n,j}] dt = c_{0,\dots,j}^n \quad (n \in \mathbb{Z}, 0 \leq j \leq K).$$

So, by Theorem 1, the conclusion of the present theorem is equivalent to the statement that the set

$$\{\exp_t[\omega_{n,0}], \exp_t[\omega_{n,0}, \omega_{n,1}], \dots, \exp_t[\omega_{n,0}, \dots, \omega_{n,K}]\} \quad (n \in \mathbb{Z}) \quad (7)$$

(viewed as a set of functions of $t \in I_K$) forms a Riesz basis for $L^2(I_K)$. This will be proved as a consequence of Theorem 2 and Lemma 1, using Lemma 2.

For $z, \lambda \in \mathbb{C}$, let $\Phi(z, \lambda) = e^{iz\lambda}$. Note that

$$\left(\frac{\partial}{\partial \lambda}\right)^j \Phi|_{(z, 0)} = (iz)^j.$$

Hence one sees, by a trivial refinement of the reasoning in [2, Theorem 2, p. 250], that the following statement holds (recall K is fixed):

For any $\varepsilon > 0$ there exists $\delta_K > 0$ such that if $\lambda_0, \dots, \lambda_K$ are distinct complex numbers with $|\lambda_j| \leq \delta_K$ ($0 \leq j \leq K$) then we have

$$|(iz)^j/j! - \exp_z[\lambda_0, \dots, \lambda_j]| < \varepsilon \quad (0 \leq j \leq K) \quad (8)$$

for any $z \in \mathbb{C}$ with $|z| \leq (K+2)\pi + 1$.

Note that, by Cauchy's estimates, the validity of (8) for $|z| \leq (K+2)\pi + 1$ implies

$$\left|\left(\frac{\partial}{\partial z}\right)^l \left(\frac{(iz)^j}{j!} - \exp_z[\lambda_0, \dots, \lambda_j]\right)\right| < \varepsilon \quad (0 \leq j \leq K, l = 0, 1, \dots)$$

for all z with $|z| \leq (K+2)\pi$. However,

$$\frac{(it)^j}{j!} e^{int} - \exp_t[\omega_{n,0}, \dots, \omega_{n,j}] = e^{int} \left(\frac{(it)^j}{j!} - \exp_t[\lambda_{n,0}, \dots, \lambda_{n,j}] \right),$$

where $n + \lambda_{n,j} = \omega_{n,j}$. Hence, by Lemma 2, we obtain the following:

For any $\varepsilon > 0$ there exists $\delta_K > 0$ such that if the $\omega_{n,j}$ are as in the statement of the theorem and $a_{-N}, \dots, a_N$ are scalars, then

$$\int_{I_K} \left| \sum_{-N}^N a_n \left(\frac{(it)^j}{j!} e^{int} - \exp_t[\omega_{n,0}, \dots, \omega_{n,j}] \right) \right|^2 dt \leq M^2 \varepsilon^2 \sum_{-N}^N |a_n|^2 \quad (0 \leq j \leq K), \quad (9)$$

where M depends only on K . It follows from (9) that for any scalars $a_{n,j}$ ($n \in \mathbb{Z}$, $0 \leq j \leq K$)

$$\begin{aligned} \int_{I_K} \left| \sum_{j=0}^K \sum_{n=-N}^N a_{n,j} \left(\frac{(it)^j}{j!} e^{int} - \exp_t[\omega_{n,0}, \dots, \omega_{n,j}] \right) \right|^2 dt \\ < 2^{(K+1)} M^2 \varepsilon^2 \sum_{j=0}^K \sum_{n=-N}^N |a_{n,j}|^2. \end{aligned} \quad (10)$$

Now, Theorem 2 shows that

$$\left\{ \frac{(it)^j}{j!} e^{int} \right\} \quad (n \in \mathbf{Z}, 0 \leq j \leq K)$$

forms a Riesz basis for $L^2(I_K)$, so (by Theorem 1) there exists $A_K > 0$ so that for any scalars a_{nj}

$$A_K^2 \sum_{j=0}^K \sum_{n=-N}^N |a_{nj}|^2 \leq \int_{I_K} \left| \sum_{j=0}^K \sum_{n=-N}^N a_{nj} \frac{(it)^j}{j!} e^{int} \right|^2 dt.$$

If M is as in (9), pick $\varepsilon > 0$ so that $2^{(K+1)} M^2 \varepsilon^2 < A_K^2$; now choose δ_K so that (9) holds with this ε , for $|n - \omega_{nj}| < \delta_K$; then (10) shows that the hypotheses of Lemma 1 are satisfied, so that the set (7) above is a Riesz basis for $L^2(I_K)$. Q.E.D.

The statement of Theorem 3 gives no information on the size of δ_K (although for any given K , a suitable value for δ_K could be obtained from a sufficiently careful examination of the proofs above, together with the proof of Theorem 2 on p. 250 of [2]). However, Theorem 3 has a corollary, the statement of which makes no mention of δ_K :

COROLLARY. Suppose $\{\omega_{nj}\}$ ($n \in \mathbf{Z}, 0 \leq j \leq K$) is a set of distinct complex numbers such that for each j ($0 \leq j \leq K$) we have $\lim_{n \rightarrow \infty} |n - \omega_{nj}| = 0$. Then the conclusion of Theorem 3 holds. (I.e., the system (5) above has a solution $f \in L^2(I_K)$ if and only if (6) holds, and if (5) has a solution it is unique.)

PROOF. As before, we need only show that the functions (7) (listed after the statement of Theorem 3) form a Riesz basis for $L^2(I_K)$, under the present hypothesis. Now, it follows easily from Theorem 1 that an independent set of vectors is a Riesz basis if it can be made into a Riesz basis by changing only finitely many terms. Thus, in view of Theorem 3, we need only show that the set (7) is independent. Hence it is enough to show that the set

$$S = \{e^{it\omega_{nj}}\} \quad (n \in \mathbf{Z}, 0 \leq j \leq K)$$

is independent, since a biorthogonal set for (7) could easily be obtained from one for S . Let $S' = S - \{e^{it\omega_{0,0}}\}$.

Theorem 3 shows that changing finitely many elements of S' results in a set which is not complete in $L^2(I_K)$. Hence, by considerations of codimension ("deficiency"), S' must be incomplete, as well. Now a result of Schwartz [7, p. 133] shows that an incomplete set of exponentials is independent; thus S' is independent. (Schwartz states his result only for ω_{nj} real, but the proof applies to general ω_{nj} .) But this shows that S is independent: Suppose not. Then, since S' is independent, $e^{it\omega_{0,0}}$ must be in the closure of the linear span of S' ; hence the incompleteness of S' implies that S is incomplete, whence the same theorem of Schwartz implies that S is independent, a contradiction. Q.E.D.

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