CYCLIC VECTORS OF WEIGHTED SHIFTS ON l^p SPACES

B. S. YADAV AND S. CHATTERJEE

ABSTRACT. Various authors have studied the existence of cyclic vectors of weighted shifts on Banach spaces (Hilbert spaces). In this paper, the existence of cyclic vectors of weighted shifts on l^p Banach spaces is exhibited under suitable conditions on the weight sequence.

1. Introduction. Let B be a Banach space and let T be an operator on B , i.e., a bounded linear transformation of B into itself. A vector $x \in B$ is called a cyclic vector of T if

$$B = \bigvee_{n=0}^{\infty} \{T^n x\},$$

the closed subspace spanned by $x, Tx, T^2x, \dots$. The existence of cyclic vectors of operators on a Banach space (Hilbert space) has been the subject of investigation by many authors; see, for example, Douglas, Shapiro and Shields [2], [3], Gellar [4], Herrero [6], Deddens, Gellar and Herrero [1], Nikolskiĭ [8], [9] and Rabindranathan [10]. The object of this note is to exhibit the existence of cyclic vectors of weighted shifts on the Banach space l^p , $1 \leq p < \infty$, under suitable restrictions on their weight sequences.

2. Let l^p , $1 < p < \infty$, be the Banach space of all complex sequences $x = \{x_0, x_1, x_2, \dots\}$ with the norm $\|x\| = (\sum_{m=0}^{\infty} |x_m|^p)^{1/p}$. We shall denote by q the conjugate index of p , i.e. the number determined by $1/p + 1/q = 1$. Let $\{w_m\}_{m=1}^{\infty}$ be a bounded sequence of nonzero complex numbers. The unique operator T on l^p defined by

$$T\{x_0, x_1, x_2, \dots\} = \{w_1x_1, w_2x_2, \dots\}$$

is called the unilateral (backward) weighted shift on l^p with the weight sequence $\{w_m\}_{m=1}^{\infty}$. We may and shall assume, without any loss of generality, that the weights w_m are positive real numbers [5].

We denote by $\{n_i\}_{i=1}^{\infty}$ a sequence of positive integers with

$$(n_{i+1} - n_i)/n_i > \xi > 1, \quad i = 1, 2, \dots,$$

and set

$$r_i^{(k)} = \sum_{m > n_i} |x_m/x_{n_i}|^k, \quad k = 1, 2, \dots \quad (1)$$

Received by the editors February 3, 1979 and, in revised form, May 30, 1979.

AMS (MOS) subject classifications (1970). Primary 47B99.

Key words and phrases. l^p Banach spaces, weighted shift operators, cyclic vectors.

© 1980 American Mathematical Society
0002-9939/80/0000-0416/\$02.25

We shall say that the sequence $\{w_m\}_{m=1}^\infty$ is of 'rth power variation' if

$$\sum_{m=1}^{\infty} |w_m^r - w_{m+1}^r| < \infty.$$

THEOREM 1. *Let T be a unilateral weighted shift on l^p with weight sequence $\{w_m\}_{m=1}^\infty$ of q th power variation and let*

$$\lambda = \lambda(k, n) = \sum_{m=0}^{\infty} \left(\frac{w_{m+k} \cdots w_{m+n}}{w_k \cdots w_n} \right)^q < \infty \quad (2)$$

for all $k (\geq 2)$ and $n (\geq k)$. Then any vector $x = \{x_m\}_{m=0}^\infty$ in l^p , such that (i) $x_{n_i} \neq 0$, $i = 1, 2, \dots$, and (ii) $r_i^{(p)} < \infty$ for sufficiently large i , is a cyclic vector of l^p .

REMARK. The conditions of the type (2) on the weight sequence $\{w_m\}_{m=1}^\infty$ and vectors x satisfying the prescribed conditions have been found useful in many other contexts also; see, for example, Kerlin and Lambert [7] and Nikolskiĭ [9].

PROOF. Let $\{e_m\}_{m=0}^\infty$ be the standard basis for l^p and let $M = \bigvee_{n=0}^\infty \{T^n x\}$. We first show that $e_0 \in M$.

By definition of T , we have $Tx = \{w_1 x_1, w_2 x_2, \dots\}$, and induction on n gives

$$T^n x = \{w_1 \cdots w_n x_n, w_2 \cdots w_{n+1} x_{n+1}, \dots\}.$$

It follows that

$$\begin{aligned} \left\| \frac{T^{n_i} x}{w_1 \cdots w_{n_i} x_{n_i}} - e_0 \right\|^p &= \sum_{m=n_i+1}^{\infty} \left(\frac{w_{m-n_i+1} \cdots w_m}{w_1 \cdots w_{n_i}} \right)^p \left| \frac{x_m}{x_{n_i}} \right|^p \\ &< \left(\sum_{m=n_i+1}^{\infty} \left(\frac{w_{m-n_i+1} \cdots w_m}{w_1 \cdots w_{n_i}} \right) \left| \frac{x_m}{x_{n_i}} \right| \right)^p \\ &= \left(\sum_{m=0}^{\infty} \left(\frac{w_{m+2} \cdots w_{m+n_i+1}}{w_1 \cdots w_{n_i}} \right) \left| \frac{x_{m+n_i+1}}{x_{n_i}} \right| \right)^p \\ &< \left(\sum_{m=0}^{\infty} \left(\frac{w_{m+2} \cdots w_{m+n_i+1}}{w_1 \cdots w_{n_i}} \right)^q \right)^{p/q} \sum_{m>n_i} \left| \frac{x_m}{x_{n_i}} \right|^p \\ &\quad \text{(by Hölder's inequality)} \\ &= \frac{1}{w_1^p} r_i^{(p)} \left(\sum_{m=0}^{\infty} \left(\frac{w_{m+2} \cdots w_{m+n_i}}{w_2 \cdots w_{n_i}} \right)^q w_{m+n_i+1}^q \right)^{p/q} \quad \text{(by (1))} \\ &= \frac{1}{w_1^p} r_i^{(p)} \left(\sum_{m=0}^{\infty} \sum_{j=0}^m \left(\frac{w_{j+2} \cdots w_{j+n_i}}{w_2 \cdots w_{n_i}} \right)^q (w_{m+n_i+1}^q - w_{m+n_i+2}^q) \right)^{p/q} \\ &\quad \text{(by Abel's transformation [11])} \\ &< \frac{1}{w_1^p} r_i^{(p)} \lambda^{p/q} \left(\sum_{m=0}^{\infty} (w_{m+n_i+1}^q - w_{m+n_i+2}^q) \right)^{p/q} \quad \text{(by (2))} \\ &< \frac{1}{w_1^p} r_i^{(p)} \lambda^{p/q} \left(\sum_{m>n_i} |w_m^q - w_{m+1}^q| \right)^{p/q} \\ &\rightarrow 0 \quad \text{as } i \rightarrow \infty. \end{aligned}$$

Thus, $e_0 \in M$.

It is immediate that, for each n , $y_n = T^n x - w_1 \dots w_n x_n e_0$ is in M . Proceeding as above, we get

$$\begin{aligned} & \left\| \frac{y_{n_i+1}}{w_2 \dots w_{n_i+1} x_{n_i+1}} - e_1 \right\|^p \\ &= \sum_{m=n_i+1}^{\infty} \left(\frac{w_{m-n_i+2} \dots w_m}{w_2 \dots w_{n_i+1}} \right)^p \left| \frac{x_m}{x_{n_i+1}} \right|^p \\ &\leq \frac{1}{w_2^p} r_{i+1}^{(p)} \lambda^{p/q} \left(\sum_{m>n_i+1} |w_m^q - w_{m+1}^q| \right)^{p/q} \\ &\rightarrow 0 \quad \text{as } i \rightarrow \infty. \end{aligned}$$

This gives $e_1 \in M$. It is clear that we can continue in this manner to exhibit that $e_m \in M$ for all m . The theorem is proved.

In our next theorem, we show that the condition of q th power variation on the weight sequence $\{w_m\}_{m=1}^{\infty}$ can be removed provided that the condition of eventual boundedness of the sequence $\{r_i\}_{i=1}^{\infty}$ is replaced by a more stringent condition.

THEOREM 2. *Let T be a unilateral weighted shift on l^p with weight sequence $\{w_m\}_{m=1}^{\infty}$. If (2) holds for all k (≥ 1) and n ($> k$), then any vector $x = \{x_m\}_{m=0}^{\infty}$ in l^p such that (i) $x_{n_i} \neq 0$, $i = 1, 2, \dots$, and (ii) $\sum_i r_i^{(p)} < \infty$, is a cyclic vector for l^p .*

PROOF. We have

$$\begin{aligned} \left\| \frac{T^{n_i} x}{w_1 \dots w_{n_i} x_{n_i}} - e_0 \right\|^p &= \sum_{m=n_i+1}^{\infty} \left(\frac{w_{m-n_i+1} \dots w_m}{w_1 \dots w_{n_i}} \right)^p \left| \frac{x_m}{x_{n_i}} \right|^p \\ &\leq \left(\sum_{m=0}^{\infty} \left(\frac{w_{m+2} \dots w_{m+n_i+1}}{w_1 \dots w_{n_i}} \right) \left| \frac{x_{m+n_i+1}}{x_{n_i}} \right| \right)^p \\ &\leq \left(\sum_{m=0}^{\infty} \left(\frac{w_{m+2} \dots w_{m+n_i+1}}{w_1 \dots w_{n_i}} \right)^{p/q} \sum_{m>n_i} \left| \frac{x_m}{x_{n_i}} \right|^p \right) \\ &\quad \text{(by Hölder's inequality)} \\ &\leq \lambda^{p/q} r_i^{(p)} \rightarrow 0 \quad \text{as } i \rightarrow \infty. \end{aligned}$$

A computation analogous to that given in the proof of Theorem 1 shows that $e_m \in \bigvee_{n=0}^{\infty} \{T^n x\}$ for all m . This completes our proof.

In the case $p = 1$, we have the following two theorems corresponding to Theorems 1 and 2.

THEOREM 3. *Let T be a unilateral weighted shift on l^1 with weight sequence $\{w_m\}_{m=1}^{\infty}$ in l^1 , and let*

$$\delta = \sup_{p, m, n} \frac{w_{p+m} \dots w_{p+n}}{w_m \dots w_n} < \infty \quad (3)$$

for all $p (> 0)$, $m (> 2)$ and $n (> m)$. Then any vector $x = \{x_m\}_{m=0}^\infty$ in l^1 , such that (i) $x_{n_i} \neq 0$, $i = 1, 2, \dots$, and (ii) $r_i^{(1)} < \infty$ for sufficiently large i , is a cyclic vector for l^1 .

THEOREM 4. Let T be a unilateral weighted shift on l^1 with weight sequence $\{w_m\}_{m=1}^\infty$. If (3) holds for all $p (> 0)$, $m (> 1)$ and $n (> m)$, then any vector $x = \{x_m\}_{m=0}^\infty$ in l^1 such that (i) $x_{n_i} \neq 0$, $i = 1, 2, \dots$, and (ii) $\sum_i r_i^{(1)} < \infty$, is a cyclic vector for l^1 .

We shall only sketch the proofs of these theorems. In case of Theorem 3, following the proof of Theorem 1, it suffices to observe that

$$\begin{aligned} \left\| \frac{T^n x}{w_1 \dots w_{n_i} x_{n_i}} - e_0 \right\| &= \sum_{m=n_i+1}^\infty \left(\frac{w_{m-n_i+1} \dots w_m}{w_1 \dots w_{n_i}} \right) \left| \frac{x_m}{x_{n_i}} \right| \\ &= \sum_{m=0}^\infty \left(\frac{w_{m+2} \dots w_{m+n_i+1}}{w_1 \dots w_{n_i}} \right) \left| \frac{x_{m+n_i+1}}{x_{n_i}} \right| \\ &= \frac{w_{n_i+1}}{w_1} \sum_{m=0}^\infty \left(\frac{w_{m+2} \dots w_{m+n_i+1}}{w_2 \dots w_{n_i+1}} \right) \left| \frac{x_{m+n_i+1}}{x_{n_i}} \right| \\ &< \frac{w_{n_i+1}}{w_1} \delta \sum_{m>n_i} \left| \frac{x_m}{x_{n_i}} \right| \\ &= \frac{\delta r_i^{(1)}}{w_1} w_{n_i+1} \rightarrow 0 \quad \text{as } i \rightarrow \infty, \end{aligned}$$

for $w_{n_i+1} \rightarrow 0$ as $i \rightarrow \infty$ and $r_i^{(1)}$ is finite for sufficiently large i .

For the proof of Theorem 4, we see that

$$\begin{aligned} \left\| \frac{T^n x}{w_1 \dots w_{n_i} x_{n_i}} - e_0 \right\| &= \sum_{m=0}^\infty \left(\frac{w_{m+2} \dots w_{m+n_i+1}}{w_1 \dots w_{n_i}} \right) \left| \frac{x_{m+n_i+1}}{x_{n_i}} \right| \\ &< \delta \sum_{m>n_i} \left| \frac{x_m}{x_{n_i}} \right| = \delta r_i^{(1)} \rightarrow 0 \quad \text{as } i \rightarrow \infty. \end{aligned}$$

Our last result investigates conditions under which a vector $x = \{x_m\}_{m=0}^\infty$ with $\{|x_m|\}_{m=0}^\infty$ monotonically decreasing serves as a cyclic vector for a weighted shift on l^p , $1 \leq p < \infty$.

THEOREM 5. Let T be a unilateral weighted shift on l^p with weight sequence $\{w_m\}_{m=1}^\infty$ of p th power variation. If for all $k (> 2)$ and $n (> k)$

$$\lambda = \lambda(k, n) = \sum_{m=0}^\infty \left(\frac{w_{m+k} \dots w_{m+n}}{w_k \dots w_n} \right)^p < \infty, \quad (4)$$

then any vector $x = \{x_m\}_{m=0}^\infty$ in l^p such that $|x_m| \geq |x_{m+1}| > 0$, $m = 0, 1, 2, \dots$, is a cyclic vector for l^p .

PROOF. Proceeding as in Theorem 1, we have

$$\begin{aligned}
 \left\| \frac{T^n x}{w_1 \cdots w_n x_n} - e_0 \right\|^p &= \sum_{m=n+1}^{\infty} \left(\frac{w_{m-n+1} \cdots w_m}{w_1 \cdots w_n} \right)^p \left| \frac{x_m}{x_n} \right|^p \\
 &\leq \sum_{m=0}^{\infty} \left(\frac{w_{m+2} \cdots w_{m+n+1}}{w_1 \cdots w_n} \right)^p \\
 &= \frac{1}{w_1^p} \sum_{m=0}^{\infty} \left(\frac{w_{m+2} \cdots w_{m+n}}{w_2 \cdots w_n} \right)^p w_{m+n+1}^p \\
 &= \frac{1}{w_1^p} \sum_{m=0}^{\infty} \sum_{j=0}^m \left(\frac{w_{j+2} \cdots w_{j+n}}{w_2 \cdots w_n} \right)^p (w_{m+n+1}^p - w_{m+n+2}^p) \\
 &\quad \text{(by Abel's transformation [11])} \\
 &\leq \frac{1}{w_1^p} \lambda \sum_{m=0}^{\infty} |w_{m+n+1}^p - w_{m+n+2}^p| \quad \text{(by (4))} \\
 &= \frac{1}{w_1^p} \lambda \sum_{m>n} |w_m^p - w_{m+1}^p| \rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

This shows that $e_0 \in \bigvee_{n=0}^{\infty} \{T^n x\}$. Now we follow exactly the line of argument as given in the proof of Theorem 1 and we are done.

The authors thank the referee for his suggestion to rewrite Theorems 3 and 4 in the present form.

REFERENCES

1. James A. Deddens, Ralph Gellar and Domingo A. Herrero, *Commutants and cyclic vectors*, Proc. Amer. Math. Soc. **43** (1974), 169–170.
2. R. G. Douglas, H. S. Shapiro and A. L. Shields, *On cyclic vectors of the backward shift*, Bull. Amer. Math. Soc. **73** (1967), 156–159.
3. ———, *Cyclic vectors and invariant subspaces for the backward shift operator*, Ann. Inst. Fourier (Grenoble) **20** (1970), fasc. 1, 37–76.
4. R. Gellar, *Cyclic vectors and parts of the spectrum of a weighted shift*, Trans. Amer. Math. Soc. **146** (1969), 69–85.
5. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967.
6. D. A. Herrero, *Eigenvectors and cyclic vectors for bilateral weighted shifts*, Rev. Un. Mat. Argentina **26** (1972), 26–41.
7. Edward Kerlin and Alan Lambert, *Strictly cyclic shifts on l^p* , Acta Sci. Math. (Szeged) **35** (1973), 87–94.
8. N. K. Nikol'skii, *The invariant subspaces of certain completely continuous operators*, Vestnik Leningrad Univ. (7) **20** (1965), 68–77. (Russian)
9. ———, *Invariant subspaces of weighted shift operators*, Mat. Sb. (N. S.) **74** (116) (1967), 172–190.
10. M. Rabindranathan, *On cyclic vectors of weighted shifts*, Proc. Amer. Math. Soc. **44** (1974), 293–299.
11. A. Zygmund, *Trigonometric series*, 2nd ed., Vol. I, Cambridge Univ. Press, New York, 1959.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI, DELHI-110007, INDIA