$\mathcal{L}^*(K)$ AND OTHER LATTICES OF RECURSIVELY ENUMERABLE SETS¹

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ABSTRACT. We study the direct product operation on lattices which are principal filters of $\mathcal{E}^*$, the lattice of r.e. sets modulo finite sets, to generate new isomorphism types of such filters and to characterize the one generated by the complete r.e. set K .

A major trend in the long term project of analyzing the lattice $\mathcal{E}$ of recursively enumerable sets and $\mathcal{E}^*$ its quotient modulo the finite sets has been the investigation of the class $\mathcal{F}$ of principal filters of $\mathcal{E}^*$, i.e. of the lattices $\mathcal{L}^*(A) = \{B \in \mathcal{E}^* \mid B^* \supseteq A\}$ for r.e. A . (Note that $A \subseteq^* B$ iff $A \triangle B$ is finite.) Of course the principal ideals of $\mathcal{E}^*$ are irrelevant since $\{B \in \mathcal{E}^* \mid B \subseteq^* A\} \cong \mathcal{E}^*$ for every $A \neq \emptyset$. The first such conscious investigations began with Myhill [1956] who defined maximal r.e. sets, i.e. sets M such that $\mathcal{L}^*(M) \cong \{0, 1\}$ (the two element Boolean algebra). Indeed the hyperhypersimple sets of Post [1944], although defined in terms of the intersection of arrays with the sets complement, also turned out to be related to this line of thought. Lachlan [1968] showed that they are precisely the r.e. sets A such that $\mathcal{L}^*(A)$ is a Boolean algebra. He was also able to completely characterize the members of $\mathcal{F}$ which are Boolean algebras as exactly the Σ_3 presentable ones.

At the other extreme one finds the r -maximal sets. These are easily seen to be equivalent to those with $\mathcal{L}^*(A)$ having no complemented elements. Classifying the isomorphism types of the r -maximal sets however seems to be a difficult open problem. The only other commonly recognized principal filter in $\mathcal{E}^*$ is the nearly ubiquitous one $-\mathcal{E}^*$ itself. Of course if A is recursive it is immediate that $\mathcal{L}^*(A) \cong \mathcal{E}^*$ but Soare [1974], [1981] has shown that this type is extremely common: If A is an r.e. infinite set and $\bar{A}$ is semilow (i.e. $\{e \mid W_e \cap \bar{A} \neq \emptyset\} < \emptyset\}$) then $\mathcal{L}^*(A) \cong \mathcal{E}^*$. This means that there are r.e. sets A in every r.e. degree with $\mathcal{L}^*(A) \cong \mathcal{E}^*$ and all low r.e. sets A (i.e., $A' <_T \emptyset'$) have this property.

Our goal here is simply to provide some additional examples of types of principal filters in $\mathcal{E}^*$. We will do this by describing some simple properties of the direct product of lattices in $\mathcal{F}$. We will then use it to generate new isomorphism types in $\mathcal{F}$. In addition these properties will enable us to make one really new identification. We will characterize the isomorphism type of $\mathcal{L}^*(K)$ by an absorption property

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with respect to products in $\mathcal{L}$. Some connections between the structure of $\mathcal{L}^*(A)$ and the degree of A will also be pointed out.

Our starting point is a simple fact from lattice theory. We work with distributive lattices with 0 and 1. Basic references are Birkhoff [1948] for lattice theory and Rogers [1967] for recursion theory. An excellent current survey of r.e. sets and degrees is Soare [1978].

LEMMA 1. $L_1 \otimes L_2 \cong L$ iff there are x_1 and x_2 in L such that $x_1 \wedge x_2 = 0$, $x_1 \vee x_2 = 1$ and $L_i \cong L(x_i)$ where $L(x_i) = \{y \in L_i \mid y \geq x_i\}$.

PROOF. The idea is just that x_1, x_2 are the images of $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$ respectively. See Birkhoff [1948, p. 26].

Our first observation is that $\mathcal{F}$ is closed under products. Consider $\mathcal{L}^*(A_1)$ and $\mathcal{L}^*(A_2)$. Let R be an infinite coinfinite recursive set with complement $\bar{R}$. We map $f_1: N \rightarrow R, f_2: N \rightarrow \bar{R}$ by one-one onto recursive maps. It is then immediate that $\mathcal{L}^*(\bar{R} \cup f_1[A_1]) \cong \mathcal{L}^*(A_1)$ and $\mathcal{L}^*(R \cup f_2[A_2]) \cong \mathcal{L}^*(A_2)$. Thus

$$\mathcal{L}^*(A_1) \otimes \mathcal{L}^*(A_2) \cong \mathcal{L}^*(\bar{R} \cup f_1[A_1]) \otimes \mathcal{L}^*(R \cup f_2[A_2])$$

but by the lemma this is just $\mathcal{L}^*(f_1[A_1] \cup f_2[A_2])$. Note that if A_1 and A_2 are simple so is $f_1[A_1] \cup f_2[A_2]$. Thus the class of principal filters generated by simple sets is also closed under direct product. Of course $\mathcal{L}^*(N) = 1$, the trivial one-element lattice, is an identity for products in $\mathcal{F}$.

We next consider $\mathcal{G}^*$ and see that it is an indecomposable idempotent.

COROLLARY 2. $\mathcal{G}^* \otimes \mathcal{G}^* \cong \mathcal{G}^*$.

PROOF. Let x_1 and x_2 be given by any infinite coinfinite recursive set and its complement.

COROLLARY 3. If $L_1 \otimes L_2 \cong \mathcal{G}^*$ then $L_i = 1$ or $\mathcal{G}^*$.

PROOF. By the lemma the L_i are isomorphic to $\mathcal{L}^*(A_i)$ for $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = N$. Thus the A_i are recursive and $\mathcal{L}^*(A_i) \cong \mathcal{G}^*$ or 1 (if one is N).

Thus products of $\mathcal{G}^*$ give no new isomorphism types in $\mathcal{F}$ and of course products of Boolean algebras are still Boolean algebras. We can however combine these two known types to generate new ones. We use a more general version of Lemma 1 to prove that one gets new types in this way.

THEOREM 4. If $\mathcal{G}^* \otimes L_1 \cong \mathcal{G}^* \otimes L_2$ then $L_1 \cong L_2$ or $L_1 \cong \mathcal{G}^* \otimes L_2$ or $L_2 = \mathcal{G}^* \otimes L_1$.

PROOF. By Theorem 7 on p. 26 of Birkhoff [1948] there are lattices $Z_1^1, Z_1^2, Z_2^1, Z_2^2$ such that $Z_1^1 \otimes Z_1^2 \cong \mathcal{G}^*, Z_1^1 \otimes Z_2^1 \cong \mathcal{G}^*, Z_2^1 \otimes Z_2^2 \cong L_1$ and $Z_1^2 \otimes Z_2^2 \cong L_2$. By Corollary 3, Z_1^1, Z_1^2 and Z_2^1 are 1 or $\mathcal{G}^*$. If $Z_1^1 \cong 1$ then $Z_1^2 = \mathcal{G}^* = Z_2^1$ and so $L_1 \cong \mathcal{G}^* \otimes Z_2^2 \cong L_2$ as required. Suppose now that $Z_1^1 \cong \mathcal{G}^*$. If $Z_2^1 \cong Z_1^2$ ($\cong 1$ or $\mathcal{G}^*$) then again $L_1 \cong L_2$ ($\cong Z_2^2$ or $\mathcal{G}^* \otimes Z_2^2$ respectively). The remaining cases are ($Z_2^1 \cong 1$ & $Z_1^2 \cong \mathcal{G}^*$) and ($Z_2^1 \cong \mathcal{G}^*$ & $Z_1^2 \cong 1$). In the first case $L_1 \cong Z_2^2$ and $L_2 \cong \mathcal{G}^* \otimes Z_2^2$ as required. The second of course gives $L_2 \cong Z_2^2$ and $L_1 \cong \mathcal{G}^* \otimes Z_2^2$. $\square$

COROLLARY 5. *If $\mathcal{B}_1 \cong \mathcal{B}_2$ are Boolean algebras in $\mathcal{F}$ then $\mathcal{G}^* \otimes \mathcal{B}_1$ and $\mathcal{G}^* \otimes \mathcal{B}_2$ are nonisomorphic elements of $\mathcal{F}$. Neither is a Boolean algebra or $\mathcal{G}^*$ but both are isomorphic to principal filters generated by simple sets.*

PROOF. As $\mathcal{G}^*$ is not a Boolean algebra $\mathcal{B}_i \cong \mathcal{G}^* \otimes \mathcal{B}_i$ and we apply the theorem. As there is a simple set A with $\mathcal{L}^*(A) \cong \mathcal{G}^*$ the $\mathcal{G}^* \otimes \mathcal{B}_i$ are isomorphic to principal filters generated by simple sets by the closure of this class under direct product.

We next want to point out a simple relation between products of elements of $\mathcal{F}$ and the degrees of the r.e. sets to which they correspond.

LEMMA 6. *If $\mathcal{L}^*(A) \cong \mathcal{L}^*(A_1) \otimes \mathcal{L}^*(A_2)$ then there are B_i with $\mathcal{L}^*(B_i) \cong \mathcal{L}^*(A_i)$ and $B_1 \oplus B_2 \equiv_T A$.*

PROOF. The elements of $\mathcal{L}^*(A)$ guaranteed by our basic lemma are now r.e. sets B_1 and B_2 with $B_1 \cap B_2 = A$, $B_1 \cup B_2 = \mathbb{N}$ and $\mathcal{L}^*(B_i) \cong \mathcal{L}^*(A_i)$. To see if $x \in B_i$ ask if $x \in A$. If so, $x \in B_i$. If not, enumerate both B_1 and B_2 until x appears in one of them. If it first appears in B_i , $x \in B_i$ and otherwise $x \notin B_i$. Of course $x \in A$ iff $x \in B_1$ and $x \in B_2$.

Thus restrictions on $\mathcal{L}^*(A)$ that push the degree of A upward are passed on by products. So we have for example

COROLLARY 7. *If $\mathcal{B} \neq 1$ is a Boolean algebra and $\mathcal{L}^*(A) \cong \mathcal{G}^* \otimes \mathcal{B}$ then A is high i.e. $\emptyset'' \equiv_T A'$.*

PROOF. By Lachlan [1968] any B with $\mathcal{L}^*(B) = \mathcal{B}$ is hyperhypersimple. Martin [1966] then shows that B must be high.

Carrying this idea to an extreme one might guess that the most complicated sets A should have $\mathcal{L}^*(A)$'s with the most factors. Indeed this gives us our characterization of $\mathcal{L}^*(K)$. ($K = \{e \mid e \in W_e\}$ is of course a 1-complete r.e. set and so in many ways the most complicated one.)

THEOREM 8. $\mathcal{L}^*(K) \cong \mathcal{L}^*(K) \otimes \mathcal{L}^*(A)$ for every r.e. A .

PROOF. Let R_1, f_1 and f_2 be as in the proof that $\mathcal{F}$ is closed under products following Lemma 1. As before we see that $\mathcal{L}^*(K) \otimes \mathcal{L}^*(A) \cong \mathcal{L}^*(f_1[K] \cup f_2[A])$. As f_1 is a recursive one-one and onto map of $\mathbb{N}$ to a recursive set R , $f_1[K]$ is also a complete set as is $f_1[K] \cup f_2[A]$. Thus by Myhill [1955] K and $f_1[K] \cup f_2[A]$ are recursively isomorphic and so

$$\mathcal{L}^*(K) \cong \mathcal{L}^*(f_1[K] \cup f_2[A]) \cong \mathcal{L}^*(K) \otimes \mathcal{L}^*(A)$$

as required. $\square$

This theorem characterizes the isomorphism type of $\mathcal{L}^*(K)$ for if $\mathcal{L}^*(B) \cong \mathcal{L}^*(B) \otimes \mathcal{L}^*(A)$ for every A then $\mathcal{L}^*(B) \cong \mathcal{L}^*(B) \otimes \mathcal{L}^*(K) \cong \mathcal{L}^*(K)$. Moreover our earlier results show that the type of $\mathcal{L}^*(K)$ is not any of the ones considered before, i.e., it is not generated as a product of lattices which are Boolean algebras or $\mathcal{G}^*$. Finally our results on degrees show that if $\mathcal{L}^*(K) \cong \mathcal{L}^*(A)$ then A is high.

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