COUNTABLY COMPACT, LOCALLY COUNTABLE T_2 -SPACES

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ABSTRACT. The results of this paper provide a simple method for constructing locally countable T_2 -spaces (not T_3) in which every infinite closed set has cardinality 2^c . The spaces are used in a variety of ways as counterexamples. One of these spaces may be considered as a countably compact version of the Katětov H -closed extension of the natural numbers.

1. Introduction. The main result in this paper (Theorem 1.1) gives a method for constructing T_2 -spaces which have small local cardinality, but which also satisfy the property that every infinite closed set has cardinality 2^c . In Corollary 1.2, we specialize this result to the Stone-Čech compactification of the natural numbers $\beta(\omega)$, and the resulting space, which may be considered as a countably compact version of the Katětov extension of the natural numbers, gives rise to the following examples.

(A) A countably compact, locally countable T_2 -space which is not D -compact for any D in $\beta(\omega) \setminus \omega$.

(B) Locally countable T_2 -spaces P and Q in which every infinite closed set has cardinality 2^c , and $P \times Q$ is not countably compact.

(C) A space which shows that property wD is not hereditary in the class of Urysohn spaces in which every point is a G_δ .

(D) A T_2 -space in which every point is a G_δ , but which has no countably infinite closed sets.

(E) A countably compact, locally countable T_2 -space in which there are no nontrivial convergent sequences. (This strengthens an example of Charles Aull [1].)

(F) A countably compact T_2 -space with density ω and weight 2^c . (This strengthens one use of the Katětov extension of the natural numbers.)

By applying Theorem 1.1 to the Gleason space of the closed unit interval we get

(G) A countably compact, H -closed, locally countable T_2 -space which is not a Baire space (this adds "locally countable" to an example of Eric van Douwen [4]).

2. Definitions and statements of results. In order to state our results and describe the examples, we need a number of definitions. We begin with those definitions needed for the statement of Theorem 1.1, and put the remaining definitions at the end of this section.

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Let (X, S) denote a set X and a topology S on X . For $A \subset X$, let $\text{cl}_S(A)$ denote the closure of A in the space (X, S) . Let ω denote the first infinite cardinal (i.e., the set of natural numbers) and $c = 2^\omega$. For a set E , let $|E|$ denote the cardinal number of E . The local cardinality of (X, S) is the smallest cardinal $\kappa \geq \omega$ such that for every $x \in X$, there exists $U \in S$ such that $x \in U$ and $|U| \leq \kappa$.

For a dense set $D \subset X$, let S_D denote the topology on X having as a base $B = \{\{x\} \cup (U \cap D) : x \in U \in S\}$. We call S_D the *Katětov extension of S with respect to D* . Clearly (1) $S \subset S_D$, (2) $D \in S_D$, and (3) $X \setminus D$ is a closed discrete set in (X, S_D) . The topology S_D is used as an aid to understanding the topology T constructed in Theorem 1.1.

Let $(\beta(\omega), S)$ be the Stone-Čech compactification of the natural numbers. Since ω is dense in $(\beta(\omega), S)$ we may consider the topology S_ω on $\beta(\omega)$. We denote this topology by K and note that $(\beta(\omega), K)$ is the Katětov extension of the natural numbers ω .

A space is called *right separated* provided there exists a well-order on X such that every initial segment in the well-order is an open set in the space.

We now can state

1.1 THEOREM. *Let (X, S) be a T_2 -space in which every infinite closed set has cardinality 2^c , and let D be a dense subset of (X, S) with $|D| < |X| = 2^c$. Then there exists a topology T on X satisfying the following conditions:*

- A. $S \subset T \subset S_D$.
- B. The local cardinality of (X, T) is $\leq |D|$.
- C. Every infinite closed subset of (X, T) has cardinality 2^c .
- D. There is a discrete subspace Z of (X, T) such that for every infinite set $F \subset X$, we have $|\text{cl}_T(F) - Z| = 2^c$, and further if $F \subset D$, then $|\text{cl}_T(F) \cap Z| = 2^c$.
- E. If D with the subspace topology induced from S is right separated, then (X, T) is right separated.

Our main use of Theorem 1.1 is the following result. We have included in the statement the values of most of the basic cardinal functions for the topology T . These are all easily calculated and left to the reader.

1.2 COROLLARY. *Let $(\beta(\omega), S)$ denote the Stone-Čech compactification of ω and $(\beta(\omega), K)$ the Katětov H -closed extension of ω . There exists a topology T on $\beta(\omega)$ such that the following hold:*

- A. $S \subset T \subset K$. Thus $(\beta(\omega), T)$ is a Urysohn space, hence strongly Hausdorff in the sense of [8], and a T_2 -space. Each $n \in \omega$ is isolated in $(\beta(\omega), T)$ and ω is dense in $(\beta(\omega), T)$. Therefore, the density, cellularity, π -weight, and depth of $(\beta(\omega), T)$ are equal to ω .
- B. The space $(\beta(\omega), T)$ is locally countable. Thus, the pseudocharacter and tightness of $(\beta(\omega), T)$ equal ω .
- C. Every infinite closed set in $(\beta(\omega), T)$ has cardinality 2^c . Thus, $(\beta(\omega), T)$ is countably compact, but has no nontrivial convergent sequences.

D. The space $(\beta(\omega), T)$ has a discrete subspace Z with $|Z| = 2^c$. Thus the weight, spread, height, and width of $(\beta(\omega), T)$ are equal to 2^c . Further, every infinite subset of ω has 2^c cluster points in Z .

E. The space $(\beta(\omega), T)$ is right separated. Thus, its Lindelöf degree is 2^c .

We now give the definitions of the main terms which are used in this paper, and refer the reader to [5] or [10] for any omitted definitions.

A space is called *locally countable* if its local cardinality is ω . The *weight* (resp. *density*) of a space is the smallest infinite cardinal number which is the cardinality of a base (resp. dense subset) of the space. A space (X, S) is called a *Urysohn space* provided for every pair of distinct points x, y in X , there exist U, V in S such that $x \in U$, $y \in V$, and $\text{cl}_S(U) \cap \text{cl}_S(V) = \emptyset$. A space is called *countably compact* (resp. *sequentially compact*) provided that every sequence in the space has a cluster point (resp. convergent subsequence). A T_2 -space is called *H-closed* if it is a closed subset of every T_2 -space in which it is embedded. A point x in a space is called a G_δ provided $\{x\}$ is the intersection of countably many neighborhoods of x , and x is called a *Hausdorff- G_δ* provided $\{x\}$ is the intersection of countably many closed neighborhoods of x . A space P is called *feebly compact* if each discrete family of open sets in P is finite. A sequence (x_n) of points in a space P is called a *discrete sequence* if for every x in P there exists a neighborhood V of x such that $|\{n: x_n \in V\}| \leq 1$. A space P has *property wD* if for every discrete sequence (x_n) in P there exists a subsequence (x_{n_i}) and a discrete family $\{V_i: i < \omega\}$ of open sets in P such that $x_{n_i} \in V_i$ if and only if $i = j$. Clearly a T_1 -space is countably compact if and only if it is feebly compact and has property wD. A space is called a *Baire space* if every countable intersection of open, dense sets is dense.

3. The examples. In order to simplify the notation in this section, let X denote the space $(\beta(\omega), T)$ given in Corollary 1.2, and let $\beta(\omega)$ denote the space $(\beta(\omega), S)$.

EXAMPLE (A). A countably compact, locally countable T_2 -space which is not D -compact for any D in $\beta(\omega) \setminus \omega$. It is an open problem to construct a countably compact, locally countable, T_3 -space which is not D -compact for any D in $\beta(\omega) \setminus \omega$. (For the definition of D -compactness, see [2] or [7].) Such spaces (which are even $T_{3\frac{1}{2}}$) have been constructed using extra axioms of set theory (see [11], [14], and [16]) but the problem is still open in ZFC. Example (A) shows that at least in the class of T_2 -spaces, we can construct such spaces within ZFC. It is known [7, Remark 2.11] that it is consistent that if P is a T_3 -space satisfying the properties of Example (A), then $|P| > c$. It is interesting, therefore, to note that the space of Example (A) can be taken to have cardinality c . To construct Example (A), take the construction of A. Bernstein [2] (which yields a countably compact subspace of $\beta(\omega)$ which is not D -compact for any D in $\beta(\omega) \setminus \omega$ and which has cardinality c) and apply it to the space X . Bernstein's construction relies mainly on the fact that every infinite closed set in $\beta(\omega)$ has cardinality 2^c ; so it will work on X .

EXAMPLE (B). Locally countable T_2 -spaces P and Q in which every infinite closed set has cardinality 2^c , and $P \times Q$ is not countably compact. This example can be compared to the following result: If P is a countably compact, locally countable

T_3 -space, and Q is any countably compact space, then $P \times Q$ is countably compact. (Proof: Every point in P is a G_δ ; so P is first countable [12], and therefore sequentially compact. It is well known that the product of a sequentially compact space with a countably compact space is countably compact [15].) Example (B) shows that T_3 cannot be weakened to T_2 in the preceding result. To construct Example (B), first recall J. Novak's well-known construction [13] of two countably compact subspaces of $\beta(\omega)$ whose product is not countably compact. This construction can be easily modified so that each of the two spaces has the property that every infinite closed set has cardinality 2^c . To get the spaces P and Q of Example (B) apply this modification of Novak's technique to the space X .

EXAMPLE (C). A space which shows that property wD is not hereditary in the class of Urysohn spaces in which every point is a G_δ . In [17], we proved that property wD is hereditary in the class of Urysohn spaces in which every point is a Hausdorff- G_δ , and we announced [17, p. 249] Example (C). We give here the construction of this example. The space X is countably compact (hence X has property wD), and is a Urysohn space in which every point is a G_δ (X is locally countable). Thus it suffices to show that X has a subspace which does not have property wD. Let Z be the discrete subspace of X given in Corollary 1.2(D). Put $W = \omega \cup Z$. Now W is feebly compact because W contains the dense set ω and every infinite subset of ω has a cluster point in $Z \subset W$ (by 1.2(D)). On the other hand, W is not countably compact since sequences in Z do not have any cluster points in W . Thus, W does not have property wD, and this completes the proof of Example (C).

An argument similar to the above shows that the Katětov extension of ω also does not have property wD. Our first attempt at Example (C) was to try to embed $(\beta(\omega), K)$ into a Urysohn space which has property wD. This cannot be done, however, because property wD is hereditary to closed subsets, and the Katětov extension of ω is H -closed. The space W , with the subspace topology induced by X , is somewhat similar to $(\beta(\omega), K)$ but they are not homeomorphic since one is H -closed and the other is not.

EXAMPLE (D). A T_2 -space in which every point is a G_δ , but which has no countably infinite closed sets. This example answers a question raised several years ago in a conversation with R. E. Hodel. Hodel has just proved [9] that if P is an infinite space in which every point is a Hausdorff- G_δ then there are $|P|^\omega$ countable, closed subsets of P . The question was raised whether the conclusion of this result still holds in " T_2 -spaces in which every point is a G_δ ". Example (D) shows that this is not the case. The space X itself has the desired properties. Note that the Katětov extension of ω cannot be used for Example (D) since it has a closed discrete subset of cardinality 2^c , hence has 2^c countable closed subsets.

In addition to the preceding examples, which we believe are new, the space X and Theorem 1.1 can be used to strengthen some known counterexamples. The following are instances of this.

EXAMPLE (E). A countably compact, locally countable T_2 -space in which there are no nontrivial convergent sequences. It is well known that in countably compact

spaces, every point which is a Hausdorff- G_δ has a countable local base, and C. Aull [1] has given an example to show that "Hausdorff- G_δ " cannot be weakened to "Hausdorff space in which every point is a G_δ " in this result. His space is a countably compact T_2 -space in which every point is a G_δ and exactly one point does not have a countable local base. It follows that Aull's space is sequentially compact. Also, it is not locally countable. To get Example (E), we may take the space X . This strengthens Aull's example in two ways. The space X is locally countable and is far from being sequentially compact (X has no nontrivial convergent sequences).

EXAMPLE (F). A countably compact T_2 -space with density ω and weight 2^c . The majority of inequalities in Juhász's book [10] hold in the class of T_2 -spaces, but some elegant ones hold only for T_3 -spaces. For example the inequality

$$\text{weight}(P) \leq 2^{\text{density}(P)}$$

holds for every T_3 -space P , but not for every T_2 -space. The Katětov extension of ω is the standard counterexample here since it has a countable dense set and weight 2^c . The space X also has these properties, and in addition is countably compact. We consider X as a countably compact version of $(\beta(\omega), K)$ in the sense that X can replace that space in a number of counterexamples.

EXAMPLE (G). A countably compact, H -closed, locally countable T_2 -space which is not a Baire space. This example requires a countable dense set of nonisolated points; so the space X cannot be used here. Eric van Douwen has given an example of a countably compact T_2 -space which is not a Baire space [4]. Example (G) is a locally countable version of van Douwen's space, and is constructed by combining Theorem 1.1 with van Douwen's idea: Let (Y, S) be a compact, separable, extremally disconnected T_2 -space having no isolated points (e.g. the Gleason space of the closed unit interval [3, 2.41]). Let D be a countable dense subset of (Y, S) , and let T be the topology on Y given by Theorem 1.1. Thus, (Y, T) is a countably compact, locally countable T_2 -space (and is H -closed by Lemma 4.1(ii)). Now (Y, T) is obviously not a Baire space since each point in D is closed and nowhere dense, and $D \in T$.

4. Proofs.

4.1 LEMMA. Let (X, S) be a space with a dense set D , and let T be a topology on X such that $S \subset T \subset S_D$.

- (i) For all $A \subset D$, $\text{cl}_S(A) = \text{cl}_T(A)$.
- (ii) If (X, S) is a compact T_2 -space, then (X, T) is H -closed.

PROOF OF (i). We need only show that $\text{cl}_S(A) = \text{cl}_{S_D}(A)$. This is half done since $S \subset S_D$ implies $\text{cl}_S(A) \supset \text{cl}_{S_D}(A)$ for any $A \subset X$. We show for $A \subset D$, that $\text{cl}_S(A) \subset \text{cl}_{S_D}(A)$. To do this, let $x \in \text{cl}_S(A) \setminus A$, and let $x \in N \in S$. Then $W = \{x\} \cup (N \cap D)$ is a typical basic neighborhood of x in (X, S_D) . Since $A \subset D$, $W \cap A = \{x\} \cap A \cup ((N \cap D) \cap A) = N \cap A$, and this set is nonempty. Thus $x \in \text{cl}_{S_D}(A)$.

PROOF OF (ii). Recall that a T_2 -space is H -closed if and only if for every filter $\mathcal{F}$ of open sets, $\bigcap \{\text{cl}(F) : F \in \mathcal{F}\} \neq \emptyset$ [5, 3.12.5]. Let $\mathcal{F} \subset T$ be a filter. Since D is dense in (X, T) , $F \cap D \neq \emptyset$ for all $F \in \mathcal{F}$. By compactness, $\bigcap \{\text{cl}_S(F \cap D) : F \in \mathcal{F}\} \neq \emptyset$. Thus, by 4.1(i), $\emptyset \neq \bigcap \{\text{cl}_T(F \cap D) : F \in \mathcal{F}\} \subset \bigcap \{\text{cl}_T(F) : F \in \mathcal{F}\}$.

PROOF OF THEOREM 1.1. By hypothesis, $|X| = 2^c$. Let $\kappa = |D|$, and well-order $X = \{u_\alpha : \alpha < 2^c\}$ such that $D = \{u_\alpha : \alpha < \kappa\}$. Let $[X]^\omega = \{H \subset X : |H| = \omega\}$. Thus, $||[X]^\omega| = (2^c)^\omega = 2^c$. Let Λ denote the set of limit ordinals $\lambda < 2^c$, and list $[X]^\omega = \{H_\lambda : \lambda \in \Lambda\}$ in such a way that for each $\lambda \in \Lambda$ we have $H_\lambda \subset \{u_\alpha : \alpha < \lambda\}$. Throughout the proof, let $\mathcal{U}(x)$ denote $\{N \in S : x \in N\}$.

The construction of T is by transfinite induction on 2^c . Put $X_0 = D$ and $T_0 = S|D$. For every $x \in X_0$, and for every $N \in \mathcal{U}(x)$ put $U(x, N) = N \cap D$. Thus, $\{U(x, N) : N \in \mathcal{U}(x)\}$ is a local base for x in the space (X_0, T_0) . This completes the first step in the induction.

Assume $\gamma < 2^c$ and that for all $\beta < \gamma$ we have defined a topology T_β on a set $X_\beta \subset X$ and sets $U(x, N)$ for all $x \in X_\beta$ and $N \in \mathcal{U}(x)$ such that the following hold for $\alpha < \beta$:

1. $|X_\beta| \leq |\beta| \cdot \kappa$ ($0 < \beta$).
2. $u_\alpha \in X_{\alpha+1}$.
3. (X_α, T_α) is an open subspace of (X_β, T_β) .
4. If $x \in X_\beta$ and $N \in \mathcal{U}(x)$ then the following hold:
 - a. $U(x, N) \subset N$ and $U(x, N) \cap D = N \cap D$,
 - b. $|U(x, N)| \leq \kappa$,
 - c. $N \subset M \in S$ implies $U(x, N) \subset U(x, M)$.
5. If $x \in X_\beta$, then $\{U(x, N) : N \in \mathcal{U}(x)\}$ is a local base for x in (X_β, T_β) .
6. $S|X_\beta \subset T_\beta \subset (S_D)|X_\beta$, where $S|X_\beta$ ($S_D|X_\beta$) denotes the subspace topology on X_β induced by S (S_D).
7. If $\lambda < \alpha$ and $\lambda \in \Lambda$, then $\text{cl}_{T_\beta}(H_\lambda) - \text{cl}_{T_\alpha}(H_\lambda) \neq \emptyset$.

We show how to construct T_γ in two cases.

Case 1. $\gamma \in \Lambda$. Put $X_\gamma = \bigcup \{X_\beta : \beta < \gamma\}$. Put $T_\gamma = \{W \subset X_\gamma : W \cap X_\beta \in T_\beta \text{ for all } \beta < \gamma\}$. There are no new $U(x, N)$ defined at this step, and it is routine to verify properties 1–7 for $\beta = \gamma$.

Case 2. $\gamma = \alpha + 1$. Since $|X_\alpha| < 2^c$, we may (by the hypothesis on (X, S)) pick a set of distinct points $\{x_\lambda^\alpha, z_\lambda^\alpha : \lambda \in \Lambda \text{ and } \lambda \leq \alpha\} \subset X \setminus X_\alpha$ such that $x_\lambda^\alpha, z_\lambda^\alpha \in \text{cl}_S(H_\lambda)$ for all $\lambda \in \Lambda, \lambda \leq \alpha$. For $N \in \mathcal{U}(x_\lambda^\alpha)$ put

$$U(x_\lambda^\alpha, N) = \{x_\lambda^\alpha\} \cup (\bigcup \{U(y, N) : y \in H_\lambda \cap N\}).$$

(This will make x_λ^α a limit point of H_λ in the topology $T_{\alpha+1}$.)

For all $N \in \mathcal{U}(z_\lambda^\alpha)$ define $U(z_\lambda^\alpha, N) = \{z_\lambda^\alpha\} \cup (D \cap N)$. (This will make z_λ^α a limit point of H_λ in the topology $T_{\alpha+1}$ if $H_\lambda \subset D$.) If $u_\alpha \notin X_\alpha \cup \{x_\lambda^\alpha, z_\lambda^\alpha : \lambda \in \Lambda \text{ and } \lambda \leq \alpha\}$, define for $N \in \mathcal{U}(u_\alpha)$, $U(u_\alpha, N) = \{u_\alpha\} \cup (D \cap N)$. (This takes care of property 2.)

Now define $X_{\alpha+1} = X_\alpha \cup \{x_\lambda^\alpha, z_\lambda^\alpha : \lambda \in \Lambda \text{ and } \lambda \leq \alpha\} \cup \{u_\alpha\}$, and $\mathcal{B} = T_\alpha \cup \{U(x, N) : x \in X_{\alpha+1} \setminus X_\alpha \text{ and } N \in \mathcal{U}(x)\}$. Note that property 4 holds for $\beta = \alpha + 1$. This can be used to show that $\mathcal{B}$ is a base for a topology on $X_{\alpha+1}$. We denote

this topology by $T_{\alpha+1}$. The verification of properties 1–7 for $\beta = \alpha + 1$ is routine, and this completes the induction. Define a topology T on $\bigcup \{X_\alpha: \alpha < 2^c\} = X$ as in the limit ordinal case. We verify (A)–(E) of Theorem 1.1 for (X, T) . Now (A) follows from 6, and (B) from 4(b). Property 7 clearly implies (C), and also (C) follows from (D). To check (D), put $Z = \{z_\lambda^\alpha: \lambda \leq \alpha, \alpha < 2^c\}$. Clearly, Z is a discrete subspace of (X, T) . By the construction, $\text{cl}_T(H_\lambda) \supset \{x_\lambda^\alpha: \alpha \geq \lambda\}$ which is disjoint from Z . Thus if F is an infinite subset of X , $|\text{cl}_T(F) - Z| = 2^c$. Similarly, if $H_\lambda \subset D$, then $\{z_\lambda^\alpha: \alpha \geq \lambda\} \subset \text{cl}_T(H_\lambda)$; so for an infinite $F \subset D$ we have $|\text{cl}_T(F) \cap Z| = 2^c$. To see that (E) holds, assume that $D = \{u_\alpha: \alpha < \kappa\}$ is a well-ordering of D such that for $\alpha < \kappa$, $\{u_\beta: \beta < \alpha\}$ is open in $(D, T|_D)$. Give $X_1 \setminus X_0$ any well-order, and combine it with the above well-order on D so that every u_α precedes every $x \in X_1 \setminus X_0$. Since each $x \in X_1 \setminus X_0$ has an open neighborhood of the form $\{x\} \cup A$ where $A \subset X_0 = D$, this shows (X_1, T_1) is right separated. Thus, proceeding by induction, we see that (X, T) is right separated. This completes the proof of Theorem 1.1.

PROOF OF COROLLARY 1.2. This follows immediately from Theorem 1.1 and the facts (i) ω is dense in $\beta(\omega)$, and (ii) every infinite closed subset of $\beta(\omega)$ has cardinality 2^c ([13] or [6, 9H]).

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