LIFTING SEMIFREE ACTIONS

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ABSTRACT. Necessary and sufficient conditions on the structure of a bundle are given, for lifting a semifree compact Lie group acting on the base to the total space.

Let X be a G -space, G a compact Lie group, and let $p: E \rightarrow X$ be a fibre bundle. We consider the problem of lifting the action of G to the total space. If X is a free G -space, it is well known that the action lifts if and only if the bundle pulls back from a bundle over the orbit space $\bar{X}$. For general G -spaces the problem is quite complicated, and although we give an obstruction theory approach in [2], [3], it seems worthwhile to give an explicit solution (Theorems A, A' of §2) in the important semifree case. If X is semifree with fixed point set F , then G acts freely on $X^0 = X - F$ and we can apply the above criterion to $E|X^0$. Likewise it is easy to describe the G -actions on $E|F$. Thus it remains to fit these two descriptions together over a neighborhood of F . For this we will use the theory and classification of equivariant bundles [1], [4].

Indeed, returning to the general lifting problem, note that if a lift exists then G acts on E through bundle automorphisms; and if say the fibre is locally compact we can give (E, p) a structure group A such that G acts on E through A -bundle automorphisms. In any case, if we assume (E, p) has a structure group A and we consider lifts of the G -action to A -bundle automorphisms, then lifts to E are equivalent to lifts to the associated principal A -bundle.

An A -bundle with G acting through A -bundle automorphisms is called an equivariant or $G - A$ bundle if it satisfies the equivariant covering homotopy property. This requires an equivariant local triviality condition [1], [4]; but if A is a closed subgroup of $GL(n, R)$ for some n , this is automatically the case (ibid).

In [2] and [4], we have generalized Wasserman's classifying space for equivariant vector bundles to numerable equivariant A -bundles, A arbitrary. The classifying space $B_G A$ is in particular a classifying space for A -bundles, but in addition $B_G A$ is a G -space and $G - A$ equivalence classes of $G - A$ bundles correspond bijectively with $[X, B_G A]_G$, the equivariant homotopy classes of equivariant maps of X to $B_G A$. That is, the pull back of a $G - A$ bundle by an equivariant map is again a $G - A$ bundle, and $B_G A$ is the base space of a universal $G - A$ bundle. Thus if $f: X \rightarrow B_G A$ is a classifying map for an A -bundle E over X , the G -action on X lifts to a $G - A$ structure on E if and only if f is homotopic to an equivariant map.

We recall the characterization of a universal $G - A$ bundle [4], and discuss some of its properties:

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A $G - A$ bundle $p: E \rightarrow X$ is universal if and only if for each closed subgroup H of G and homomorphism $\rho: H \rightarrow A$,

(1) E has a fixed point under the H -action $z \rightarrow hz\rho(h)^{-1}$, $h \in H$, $z \in E$.

(2) For any fixed point $z_0 \in E$ under the above action, E is H contractible to z_0 .

A universal $G - A$ bundle will be denoted $\pi_G A: E_G A \rightarrow B_G A$. First note that $x \in B_G A$ is fixed under H if and only if for any $z \in \pi^{-1}(x)$, $hz = z\rho_z(h)$, $\rho_z: H \rightarrow A$ a homomorphism. Further, $\rho_{za} = a^{-1}\rho_z a$, $a \in A$. Thus each $x \in (B_G A)^H$ can be assigned a well-defined A -equivalence class (ρ) of homomorphisms $\rho: H \rightarrow A$. Now $x \in (\rho)$ if and only if there is a $z \in \pi^{-1}(x)$ such that $hz\rho(h)^{-1} = z$. By the above characterization, the fixed point set F^ρ of E under this action is contractible. Further, if $z \in F^\rho$, $za \in F^\rho$ if and only if $a \in A^\rho = \text{centralizer of } \rho(H) \text{ in } A$. Thus F^ρ is an A^ρ -subbundle and indeed a universal A^ρ -bundle, since it is contractible. Thus $\pi(F^\rho) = B(A^\rho)$ and $(B_G A)^H \simeq \coprod_{(\rho)} B(A^\rho)$; i.e., one ρ from each A equivalence class. Write $B^{(\rho)} = \pi(F^\rho)$ and $E^{(\rho)} = \pi^{-1}(B^{(\rho)})$. Then $E^{(\rho)} = E(A^\rho) \times_{A^\rho} A$, as an A -bundle.

1. Bundles over a free G -space. If X is a free G -space and $\bar{X} = X/G$, we have the well-known

LEMMA 1.1. *There is a bijective correspondence between $G - A$ equivalence classes of $G - A$ bundles over X and A equivalence classes of A -bundles over $\bar{X}$.*

In fact, if P is a $G - A$ bundle over X , $\bar{P} = P/G$ is the corresponding A -bundle over $\bar{X}$. If $q: X \rightarrow \bar{X}$ is the quotient map and Q is an A -bundle over $\bar{X}$ considered as a $G - A$ bundle with trivial G -action, then the induced bundle q^*Q has a natural $G - A$ structure since q is equivariant. These constructions preserve equivalence classes and are inverse to each other.

Now suppose we start with a $G - A$ bundle Q over the trivial G -space $\bar{X}$ but with nontrivial G -action on Q . Then q^*Q is again a $G - A$ bundle over X but in general $\overline{q^*Q}$ is not A -equivalent to Q . To see just how $\overline{q^*Q}$ is related to Q , first note that Q is classified by an equivariant map $f: \bar{X} \rightarrow B_G A$, and $f(\bar{X}) \subset (B_G A)^G = \coprod_i B^{(\rho_i)}$, $\rho_i: G \rightarrow A$. Assuming $\bar{X}$ is connected, $f(\bar{X})$ is contained in a single component of $(B_G A)^G$, say $B^{(\rho)}$, $\rho: G \rightarrow A$. Since $E^{(\rho)} = EA^\rho \times_{A^\rho} A$, $Q = Q_0 \times_{A^\rho} A$, Q_0 an A^ρ -bundle, with $g[z_0, a] = [z_0, \rho(g)a]$. Considering X as a principal G -bundle over $\bar{X}$ and letting $\gamma: G \times A^\rho \rightarrow A$ be the homomorphism $\gamma(g, a_0) = \rho(g)a_0$, we can form the γ -product, $X \otimes_\gamma Q_0$, an A -bundle over $\bar{X}$. That is, $X \times Q_0$ is a $G \times A^\rho$ bundle over $\bar{X} \times \bar{X}$ and $X \otimes_\gamma Q_0$ is the pull back via $\Delta: \bar{X} \rightarrow \bar{X} \times \bar{X}$ of the associated bundle with group A . Then by a simple check of definitions we get

LEMMA 1.2. $q^*Q \simeq q^*(X \otimes_\gamma Q_0)$ as $G - A$ bundles (or equivalently, $\overline{q^*Q} \simeq X \otimes_\gamma Q_0$ as A -bundles).

PROOF.

$q^*Q = q^*Q_0 \times_{A^\rho} A = \{[(x, z_0), a] | \bar{x} = p(z_0) \text{ and } [(x, z_0 a_0), a] = [(x, z_0), a_0 a]\}$
with G -action, $g[(x, z_0), a] = [(gx, z_0), \rho(g)a]$.

$$\begin{aligned}
q^*(X \otimes_{\gamma} Q_0) &= q^*X \otimes_{\gamma} q^*Q_0 \\
&= \{ [(x, x'), (x'', z_0), a] | \bar{x} = \bar{x}' = p(z_0) \text{ and} \\
&\quad [(x, g^{-1}x'), (x'', z_0a_0), a] = [(x, x'), (x'', z_0), \rho(g)a_0a] \}
\end{aligned}$$

with G -action $g[(x, x'), (x'', z_0), a] = [(gx, x'), (gx'', z_0), a]$. Note that since q^*X has a natural cross-section there is a natural injection $i_2: q^*Q_0 \rightarrow q^*X \otimes_{\gamma} q^*Q_0$; namely,

$$i_2(x, z_0) = [(x, x), (x, z_0), 1].$$

Hence we can define a unique A -bundle equivalence $\varphi: q^*Q_0 \times_{A^e} A \rightarrow q^*X \otimes_{\gamma} q^*Q_0$ extending i_2 ; i.e.,

$$\varphi[(x, z_0), a] = [(x, x), (x, z_0), a].$$

It is easy to check that φ is well defined and a $G - A$ bundle equivalence.

ADDENDUM. There is a *unique* $G - A$ bundle equivalence, indeed a unique A -bundle equivalence φ extending the identity map in the subspace q^*Q_0 :

$$\begin{array}{ccc}
& q^*Q_0 & \\
i_0 \swarrow & & \searrow i_2 \\
q^*Q & \xrightarrow{\varphi} & q^*(X \otimes_{\gamma} Q_0)
\end{array}$$

There are classifying map versions of 1.1 and 1.2:

LEMMA 1.1'. If X is a free G -space, $[X, B_G A]_G \simeq [\bar{X}, BA]$. Further, if we identify BA with $B^{(\rho_0)} \subset (B_G A)^G$, then the correspondence is given by $\bar{f}: \bar{X} \rightarrow B^{(\rho_0)}$ going to $\bar{j}_0 \bar{f} q: X \rightarrow B^{(\rho_0)} \subset B_G A$. (ρ_0 is the trivial homomorphism.)

PROOF. As shown in [2], [5], if X is a free G -space and Y an arbitrary G -space, equivariant maps of X to Y correspond to cross-sections of the fibration $X \times_G Y \rightarrow \bar{X}$ associated to $g: X \rightarrow \bar{X}$. If $Y_0 \subset Y^G$ is a deformation retract of Y , then $\bar{X} \times Y_0 = X \times_G Y_0 \hookrightarrow X \times_G Y$ is a f.h.e. Thus the space of equivariant maps of X to Y has those of the form $\bar{f} q, \bar{f}: \bar{X} \rightarrow Y_0$, as deformation retract. Taking $Y = B_G A$ and $Y_0 = B^{(\rho_0)}$, the result follows.

The lemma below is equivalent to 1.2.

LEMMA 1.2'.

$$\begin{array}{ccc}
EG \times B^{(\rho)} & \xrightarrow{pr_2} & B^{(\rho)} \\
& & \searrow \bar{j} \\
q \times 1 \downarrow & & B_G A \\
& & \nearrow \bar{j}_0 \\
BG \times B^{(\rho)} & \xrightarrow{\bar{\gamma}_0} & B^{(\rho_0)}
\end{array}$$

commutes up to equivariant homotopy.

PROOF. If we take $X = EG \times B^{(\rho)}$ and $Q = BG \times E^{(\rho)}$, then $\bar{j}pr_2$ classifies q^*Q and $\bar{j}_0 \bar{\gamma}_0(q \times 1)$ classifies $q^*(X \oplus_{\gamma} Q_0)$. Thus (1.2) implies (1.2').

Conversely, if $\bar{f}: \bar{X} \rightarrow B^{(p)}$ classifies Q_0 and $\lambda: X \rightarrow EG$ is any G -map, then $\bar{f}\bar{f}q = \bar{f}pr_2(\lambda, \bar{f}q)$ induces q^*Q . By (1.2'), this is equivariantly homotopic to $\bar{f}_0\bar{\gamma}_0(q\lambda, \bar{f}q) = \bar{f}_0\bar{\gamma}_0(\bar{\lambda}, \bar{f})q$, which induces $q^*(X \otimes_\gamma Q_0)$. Hence (1.2') implies (1.2).

2. Bundles over a semifree G -space. Let X be a semifree G -space. Let $F = \coprod_i F_i$ be the decomposition of the fixed point set into connected components. We assume F is an equivariant neighborhood deformation retract. (This holds for example if X is a smooth G -manifold or more generally a G -CW complex.) Let $\{N_i\}$ be disjoint closed invariant neighborhoods of the F_i and $r_i: N_i \rightarrow F_i$ equivariant (deformation) retracts. Let $N = \coprod_i N_i$, and N' a smaller open invariant neighborhood of F ; i.e., $\text{cl}(N') \subseteq \text{Int } N$. Write $X^0 = X - N'$, $N^0 = N - N'$ and $N_i^0 = N_i - N'$. Also write $\bar{X}^0, \bar{N}^0, \bar{N}_i^0$ for the orbit spaces and $q_0: X^0 \rightarrow \bar{X}^0$, etc., for the quotient maps. Also let $\bar{\lambda}_i^0: \bar{N}_i^0 \rightarrow BG$ be a classifying map for the $q_i^0: N_i^0 \rightarrow \bar{N}_i^0$, $\lambda_i^0: N_i^0 \rightarrow EG$ the covering bundle map.

THEOREM A. *The G -action on X lifts to a $G - A$ structure on an A -bundle E over X if and only if the bundle has the form:*

- (1) $E|N_i = r_i^*P_i \times_{A^{p_i}} A$, P_i an A^{p_i} -bundle over F_i , some $\rho_i: G \rightarrow A$.
- (2)(a) $E|X^0 = q^*E^0$, E^0 an A -bundle over $\bar{X}^0$, with
 (b) $E^0|\bar{N}_i^0 = N_i^0 \otimes_{\gamma_i} \bar{r}_i^{0*}P_i$, $\gamma_i: G \times A^{p_i} \rightarrow A$, $\gamma_i(g, a) = \rho_i(g)a$.
- (3) $E = E|N \cup E|X^0$, where over N_i^0 we identify $q^*(N_i^0 \otimes_{\gamma_i} \bar{r}_i^{0*}P_i)$ with $r_i^{0*}P_i \times_{A^{p_i}} A = q^*r_i^{0*}P_i \times_{A^{p_i}} A$ by the unique A -isomorphism extending the identity map in $q^*r_i^{0*}P_i$.

PROOF. (a) Suppose we have a $G - A$ bundle E over X .

(1) Since the classifying map $f: X \rightarrow B_G A$ sends F to $(B_G A)^G$, each F_i maps to some $B^{(p_i)}$, $\rho_i: G \rightarrow A$. So $E|F_i$ reduces to an A^{p_i} -bundle P_i . Since $r_i: N_i \rightarrow F_i$ is an equivariant deformation retract

$$E|N_i = r_i^*E|F_i = r_i^*P_i \times_{A^{p_i}} A = q^*(\bar{r}_i^*P_i \times_{A^{p_i}} A).$$

(2) Since X^0 is a free G -space, Lemma 1.1 says $E|X^0 = q^*\bar{E}^0$, where $\bar{E}^0 = (E|X^0)/G$.

Since $E|N_i^0 = q^*(\bar{r}_i^{0*}P_i \times_{A^{p_i}} A)$ we may identify $\bar{E}^0|\bar{N}_i^0$ with $N_i^0 \otimes_{\gamma_i} \bar{r}_i^{0*}P_i$ by Lemma 1.2.

By Lemma 1.2, Addendum, this means

(3) $q^*(N_i^0 \otimes_{\gamma_i} \bar{r}_i^{0*}P_i)$ is identified to $q^*\bar{r}_i^{0*}P_i \times_{A^{p_i}} A$ by the unique A -isomorphism extending the identity map on $q^*\bar{r}_i^{0*}P_i$.

(b) Suppose we have an A -bundle E satisfying (1), (2), (3) of Theorem A.

(1) Give $E|F_i = P_i \times_{A^{p_i}} A$ the G -action, $g[z, a] = [z, \rho_i(g)a]$. Give $E|N_i = r_i^*(P_i \times_{A^{p_i}} A)$ the induced G -action.

(2) Give $E|X^0$ the induced G -action from the trivial G -action on $\bar{E}^0$.

(3) The identification of $E|N$ with $E|X^0$ over N_i^0 is then a $G - A$ equivalence by condition (3) (see 1.2, Addendum) and thus $E = E|N \cup E|X^0$ has a $G - A$ structure.

Theorem A may be reformulated in terms of classifying maps. For this purpose it will be convenient to replace $X = N \cup_{N^0} X^0$ by $\hat{X} = N \cup_{1_{N^0}} N^0 \times I \cup_{1_{N^0}} X^0$ the

space of the same G -homotopy type obtained by thickening N^0 .

THEOREM A'. *A map f of $\hat{X}$ into BA defines an A -bundle over $\hat{X}$ which admits a $G - A$ structure if and only if for some collection of homomorphisms $\rho_i: G \rightarrow A$, f has (up to homotopy) the form:*

$$f|N_i: N_i \xrightarrow{r_i} F_i \xrightarrow{h_i} BA^{\rho_i} \xrightarrow{\bar{j}_i} BA, \quad (1)$$

$\bar{j}_i$ induced by the inclusion $j_i: A^{\rho_i} \rightarrow A$.

$$f|X^0: X^0 \xrightarrow{q^0} \overline{X^0} \xrightarrow{\bar{f}^0} BA, \quad (2)$$

with

$$\bar{f}^0|N_i^0: N_i^0 \xrightarrow{(\bar{\lambda}_i, h_i r_i^0)} BG \times BA^{\rho_i} \xrightarrow{\bar{\gamma}_i} BA,$$

$\bar{\gamma}_i$ induced by γ_i , $\gamma_i(g, a) = \rho_i(g)a$.

(3) $f|N_i^0 \times I$ is the homotopy from $\bar{f}^0 q^0|N_i^0$ to $\bar{j}_i h_i r_i^0 = \bar{j}_i h_i \bar{r}_i^0 q^0$ derived by deforming $\bar{\lambda}_i q_i^0 = q \lambda_i^0: N_i^0 \rightarrow BG$ to a constant map.

REMARK. In (3) the map

$$\bar{f}^0 q^0|N_i^0: N_i^0 \xrightarrow{(\lambda_i^0, h_i r_i^0)} EG \times BA^{\rho_i} \xrightarrow{q \times 1} BG \times BA^{\rho_i} \xrightarrow{\bar{\gamma}_i} BA$$

is deformed to $\bar{\gamma}_0 \circ \bar{i}_0 \circ h_i r_i^0$, $i_0: A^{\rho_i} \rightarrow G \times A^{\rho_i}$, $i_0(a) = (1, a)$, by the contractibility of EG . But $j_i = \gamma_i \circ i_0$.

Theorem A' follows from Lemmas 1.1', 1.2', and the structure of $B_G A$ described above.

EXAMPLE. *Complex n -plane bundles over a semifree space X . Then $A = U(n)$ and any $\rho: G \rightarrow U(n)$ is equivalent to a direct sum of irreducible representations; say $\rho = n_1 \rho_1 \oplus \cdots \oplus n_k \rho_k$. The centralizer of ρ is the subgroup $A^\rho = (U(n_1) \otimes I_{d_1}) \oplus \cdots \oplus (U(n_k) \otimes I_{d_k})$, where $d_j = \dim \rho_j$.*

Thus if F is a component of the fixed points of G on X , and E is a complex vector bundle over X such that the action lifts, then for some $\rho: G \rightarrow U(n)$, as above

(a) $E|F = (E_1 \times \varepsilon_{d_1}) \oplus \cdots \oplus (E_k \times \varepsilon_{d_k}) = \bigoplus_i d_i E_i$, E_i an n_i -dimensional vector bundle.

(b) $\bar{E}|\bar{N}_0 = E_1 \otimes E'_1 \oplus \cdots \oplus E_k \otimes E'_k$, $E'_i = N_0 \times_{\rho_i} \mathbb{C}^{d_i}$, $N_0 = N - F$, N an invariant neighborhood of F .

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