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ON CONJUGACY CLASSES OF ELEMENTS OF FINITE ORDER IN COMPACT OR COMPLEX SEMISIMPLE LIE GROUPS

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ABSTRACT. If K is a connected compact Lie group with simple Lie algebra and if k is an integer relatively prime to the order of the Weyl group W of K then the number $\nu(K, k)$ of conjugacy classes of K consisting of elements x satisfying $x^k = 1$ is given by

$$\nu(K, k) = \prod_{i=1}^{l} \frac{m_i + k}{m_i + 1},$$

where l is the rank of K and $m_1, \dots, m_l$ are the exponents of W. If G is the complexification of K then we have $\nu(G, k) = \nu(K, k)$ without any restriction on k.

Results and proofs. If G is a group and k a positive integer we write $G(k) = \{x \in G | x^k = 1\}$. We denote by $\nu(G, k)$ the number of conjugacy classes of G contained in $G(k)$. (In our cases $\nu(G, k)$ will be finite.)

LEMMA 1. *Let Z_1 be a finite subgroup of the center of G. If k and $|Z_1|$ are relatively prime then the canonical map $G \to G/Z_1$ induces a bijection $G(k) \to (G/Z_1)(k)$ and $\nu(G, k) = \nu(G/Z_1, k)$.*

PROOF. Let $x, y \in G(k)$ and assume that $xZ_1 = yZ_1$. Then $y = xz$ for some $z \in Z_1$. Hence $1 = y^k = (xz)^k = z^k$. Since k and $|Z_1|$ are relatively prime, we have $z = 1$, and so $x = y$.

Now let $x \in G$ be such that $xZ_1 \in (G/Z_1)(k)$, i.e., $x^k \in Z_1$. Since k and $|Z_1|$ are relatively prime, there exists $z \in Z_1$ such that $x^k = z^k$. Then $y = xz^{-1} \in G(k)$ and $yZ_1 = xZ_1$. Thus we have shown that $G(k) \to (G/Z_1)(k)$ is a bijection. The second assertion now follows immediately.

From now on let G be a connected complex semisimple Lie group, g its Lie algebra, l the rank of g, h its Cartan subalgebra, H the corresponding Cartan subgroup of G, N the normalizer of H in G, and $W = N/H$ the Weyl group of (g, h). By P we denote the group of weights of (g, h) and by Q the subgroup of P of radical weights. Both P and Q are free abelian groups of rank l, and Q is generated by the root system Σ of (g, h). The group H is an algebraic torus, i.e., H is isomorphic to the product of l copies of the group C^* of nonzero complex numbers. The exponential map $\exp_H$: $h \to H$ is a surjective homomorphism. The

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kernel L_G of the homomorphism $f\colon h \to H$ defined by $f(X) = \exp_H(2\pi i X)$ is also a free abelian group of rank l which is generated by a basis of h (considered as a complex vector space). Let P_G be the subgroup of the dual space h^* consisting of the elements α such that $\alpha(X)$ is an integer for all $X \in L_G$. Then we have $Q \subset P_G \subset P$. We fix a positive integer k. The Weyl group W acts on h and h^* and P_G is stable under this action. Hence W also acts on the finite abelian group P_G/kP_G. (Note that this abelian group is a direct product of l cyclic groups of order k.) We recall also that the order e of P/Q is called the connection index of g (or W).

Our first result is the following.

THEOREM 2. *Let K be a maximal compact subgroup of G. Then $\nu(G, k) = \nu(K, k)$ and this number is also equal to the number of orbits of W in P_G/kP_G.*

PROOF. Let T be the (unique) maximal compact subgroup of H. Since all maximal compact subgroups of G are conjugate in G we may assume that $T \subset K$. Then T is a maximal torus of K. Every $x \in G(k)$ is conjugate to some $y \in K$. Since K is connected, y is conjugate in K to some element of T. Hence every conjugacy class of G which is contained in $G(k)$ meets $H(k)$. On the other hand two elements of H are conjugate in G iff they belong to the same orbit of W in H. This shows that the inclusion map $H(k) \to G(k)$ induces a bijection from the set of W-orbits in $H(k)$ to the set of G-conjugacy classes contained in $G(k)$. Similarly, the inclusion map $H(k) = T(k) \to K(k)$ induces a bijection from the set of W-orbits in $H(k)$ to the set of K-conjugacy classes contained in $K(k)$. In particular, we have $\nu(G, K) = \nu(K, k)$.

The epimorphism $f\colon h \to H$ induces a bijection between the set of W-orbits in $(k^{-1}L_G)/L_G$ and the set of W-orbits in $H(k)$. Finally, by duality, the number of W-orbits in $(k^{-1}L_G)/L_G$ is equal to the number of W-orbits in P_G/kP_G. This completes the proof.

Now let us assume that k and $|W|$ are relatively prime. Let Z be the center of G. Then it is easy to check that every prime divisor of $|Z|$ also divides $|W|$. Hence k and $|Z|$ are also relatively prime. By Lemma 1 we have then $\nu(G, k) = \nu(G/Z, k)$. Thus we may assume that G is the adjoint group. Then G is a product of simple complex Lie groups $G_1, \ldots, G_r$. Consequently we have

$$\nu(G, k) = \prod_{i=1}^{r} \nu(G_i, k).$$

This reduces the problem of computing $\nu(G, k)$ to the case when G is a simple complex Lie group (with trivial center). In that case the answer is given in the following theorem.

THEOREM 3. *Assume that G is the adjoint group, g is simple, and that k and W are relatively prime. Then we have*

$$\nu(G, k) = \prod_{i=1}^{l} \frac{m_i + k}{m_i + 1}$$

where $m_1, \ldots, m_l$ are the exponents of W. (See [**1**, *p.* 118].)

PROOF. In this case we have $P_G = Q$. By Theorem 2, $\nu(G, k)$ is equal to the number of orbits of W in Q/kQ.

For each root $\alpha \in \Sigma$ let $s_\alpha \in W$ be the corresponding reflection. A root system $\Sigma_1 \subset \Sigma$ is closed in Σ if $\alpha, \beta \in \Sigma_1$ and $\alpha + \beta \in \Sigma$ imply that $\alpha + \beta \in \Sigma_1$. A subgroup $W_1 \subset W$ is called a Weyl subgroup if there exists a closed subsystem $\Sigma_1 \subset \Sigma$ such that W_1 is generated by the reflections s_α for all $\alpha \in \Sigma_1$. Then the set of all $\alpha \in \Sigma$ such that $s_\alpha \in W_1$ is a closed subsystem containing Σ_1. Hence without loss of generality we may assume that $\Sigma_1 = \{\alpha \in \Sigma | s_\alpha \in W_1\}$.

Fix $w \in W$ and let W_1 be a minimal Weyl subgroup of W containing w. Define Σ_1 as above and let

$$\Sigma_1 = \Sigma_{11} \cup \cdots \cup \Sigma_{1r}$$

be the decomposition of Σ_1 into irreducible root systems. Then the real vector subspace of h^* spanned by Σ_1 admits a direct decomposition

$$V_1 = V_{11} + \cdots + V_{1r},$$

where V_{1i} is spanned by Σ_{1i}. This leads to the corresponding direct decomposition of the group W_1:

$$W_1 = W_{11} \times \cdots \times W_{1r},$$

where W_{1i} is generated by the reflections s_α for $\alpha \in \Sigma_{1i}$. Let $w = w_1 \ldots w_r$ be the corresponding decomposition of the element w. By minimality of W_1, the element $w_i \in W_{1i}$ $(i = 1, \ldots, r)$ is not contained in any proper Weyl subgroup of W_{1i}. By [**4**, Corollary 8.3] we have $\det(w_i - 1) = \pm e_i$ where e_i is the connection index of W_{1i}, and w_i is considered as acting in V_{1i}. Hence

$$\det(w|_{V_1} - 1) = \pm e_1 \cdots e_r.$$

Since e_i divides $|W_{1i}|$, and the latter divides $|W|$, it follows that k and the above determinant are relatively prime.

Let m be the multiplicity of the eigenvalue 1 of w. Thus $\dim V_1 = l - m$. By [**2**, Theorem III.12, p. 50] there exists a basis of Q with respect to which the matrix of w is an integral l by l matrix of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A is an upper triangular m by m matrix with ones on the diagonal. Since w has finite order, A must be the identity matrix. Since

$$\det(C - I) = \det(w|_{V_1} - 1),$$

the matrix $C - I$ is invertible when considered as a matrix over the residue ring $\mathbf{Z}/k\mathbf{Z}$. Consequently, the number of elements of Q/kQ fixed by w is equal to k^m.

Let g_m be the number of elements $w \in W$ such that 1 is an eigenvalue of w of multiplicity m. By a theorem of Solomon [**3**] we have the identity

$$(m_1 + t) \cdots (m_l + t) = g_0 + g_1 t + \cdots + g_l t^l.$$

Hence $|W| = (m_1 + 1) \ldots (m_l + 1)$ and the number of orbits of W in Q/kQ is equal to

$$\frac{1}{|W|} \sum_{m=0}^{l} g_m k^m = \prod_{i=1}^{l} \frac{m_i + k}{m_i + 1}.$$

This completes the proof of the theorem.

REMARKS. 1. The equality $\nu(G, k) = \nu(K, k)$ from Theorem 1 is in fact valid for any compact Lie group K (not necessarily semisimple nor connected) and its complexification G.

2. If $\nu(G, d)$ is known for all divisors d of k then by using Inclusion-Exclusion Principle one can easily compute the number of conjugacy classes of G consisting of elements of order k.

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