A SHORT PROOF OF A GENERATING FUNCTION FOR JACOBI POLYNOMIALS

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ABSTRACT. A short proof is given for Bailey's bilinear generating function for Jacobi polynomials. It depends only upon the orthogonality relation for Jacobi polynomials and a quadratic transformation for a hypergeometric series. A q -analog is also stated.

Bailey [4, equation (2.1)] found the following bilinear generating function for Jacobi polynomials:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{n! (\alpha + \beta + 1)_n}{(\alpha + 1)_n (\beta + 1)_n} P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y) t^n \\ = (1+t)^{-\alpha-\beta-1} F_4((\alpha + \beta + 1)/2, (\alpha + \beta + 2)/2; \alpha + 1, \beta + 1; z, Z) \\ = (1+t)^{-\alpha-\beta-1} \sum_{n, m=0}^{\infty} \frac{(\alpha + \beta + 1)_{2(m+n)}}{n! m! (\alpha + 1)_n (\beta + 1)_m} \left[\frac{t}{4(1+t)^2} \right]^{m+n} \\ \cdot (1+x)^m (1+y)^m (1-x)^n (1-y)^n, \end{aligned} \quad (1)$$

where $z = t(1+x)(1+y)/(1+t)^2$, $Z = t(1-x)(1-y)/(1+t)^2$, and $(a)_n = \Gamma(a+n)/\Gamma(a)$. He used Watson's transformation, which expresses an F_4 as the product of two ${}_2F_1$'s. In this paper we give a short proof of (1) which uses only the orthogonality of Jacobi polynomials and a quadratic transformation for a generalized hypergeometric series.

The Jacobi polynomials $\{P_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty}$ are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha, \beta > -1$. We take the usual normalization, $P_n^{(\alpha, \beta)}(1) = (\alpha + 1)_n / n!$, so that $P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x)$. Askey [3] has shown how to obtain Jacobi's generating function for $P_n^{(\alpha, \beta)}(x)$ from the orthogonality relation. The same idea works for the bilinear generating function.

Let $G(x, y, t) = \sum_{n=0}^{\infty} a_n(x, y) t^n$ be the right-hand side of (1). It is clear that $a_n(x, y)$ is a polynomial of degree at most n in both x and y . For any polynomials $q_k(x)$ and $r_l(y)$ of degrees k and l , consider

$$I(t) = \int_{-1}^1 \int_{-1}^1 G(x, y, t) (1-x)^\alpha (1+x)^\beta q_k(x) (1-y)^\alpha (1+y)^\beta r_l(y) dx dy.$$

A necessary condition for $a_n(x, y) = b_n P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y)$, $b_n \neq 0$, is that $I(t)$ is a polynomial in t of degree $\min(k, l)$. This follows immediately from the orthogonality relation for $P_n^{(\alpha, \beta)}$. However this condition is also sufficient. To see this, let

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$$a_n(x, y) = \sum_{i,j=0}^n b_{ij}^n P_i^{(\alpha, \beta)}(x) P_j^{(\alpha, \beta)}(y).$$

By successively choosing $k, l = 0, 1, \dots, n$ we have that $b_{ij}^n = 0$ unless $i = j = n$. This proof works for any set of orthogonal polynomials with a positive weight function $da(x)$ which has an infinite number of points of increase.

To prove (1) we put $q_k(x) = (1+x)^k$ and $r_l(y) = (1-y)^l$. The integrals are evaluated by beta functions to obtain

$$I(t) = A(1+t)^{-\alpha-\beta-1} \sum_{m,n=0}^{\infty} \frac{(\alpha+\beta+1)_{2(m+n)} (\beta+k+1)_m (\alpha+l+1)_n}{m!n! (\alpha+\beta+k+2)_{m+n} (\alpha+\beta+l+2)_{m+n}} \cdot \left[\frac{t}{(1+t)^2} \right]^{n+m} \quad (2)$$

where $A \neq 0$ is independent of t . Putting $n = N - m$, and summing the resulting m sum by Vandermonde's theorem [5, p. 3], we have

$$I(t) = A(1+t)^{-\alpha-\beta-1} \sum_{N=0}^{\infty} \frac{(\alpha+\beta+1)_{2N} (\alpha+\beta+l+k+2)_N}{N! (\alpha+\beta+k+2)_N (\alpha+\beta+l+2)_N} \cdot \left[\frac{t}{(1+t)^2} \right]^N \\ = A(1+t)^{-\alpha-\beta-1} {}_3F_2 \left(\begin{matrix} (\alpha+\beta+1)/2, & (\alpha+\beta+2)/2, \\ & \alpha+\beta+k+2, \end{matrix} \middle| \frac{4t}{(1+t)^2} \right). \quad (3)$$

However there is a quadratic transformation for a well-poised ${}_3F_2$ [5, p. 97]

$$(1+t)^{-a} {}_3F_2 \left(\begin{matrix} a/2, & (a+1)/2, & 1+a-b-c \\ & 1+a-b, & 1+a-c \end{matrix} \middle| \frac{4t}{(1+t)^2} \right) \\ = {}_3F_2 \left(\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c, & -t \end{matrix} \right), \quad (4)$$

Thus we have for $a = \alpha + \beta + 1, b = -l, c = -k$,

$$I(t) = A {}_3F_2 \left(\begin{matrix} -k, & -l, & \alpha+\beta+1 \\ \alpha+\beta+2+k, & \alpha+\beta+2+l, & -t \end{matrix} \right),$$

which is clearly a polynomial in t of degree $\min(k, l)$. To evaluate the constants b_n put $x = 1$ and $y = -1$ to obtain

$$\sum_{n=0}^{\infty} b_n P_n^{(\alpha, \beta)}(1) P_n^{(\alpha, \beta)}(-1) t^n = (1+t)^{-\alpha-\beta-1}.$$

Since $P_n^{(\alpha, \beta)}(1) = (\alpha+1)_n/n!$ and $P_n^{(\alpha, \beta)}(-1) = (\beta+1)_n(-1)^n/n!$ we obtain (1).

Unfortunately we were not directly led to (1). However, we can write down a q -analog of equation (1) by this technique. We take the little q -Jacobi polynomials $P_n(x, a, b; q)$ as defined in [1, equation (3.1)]. Carlitz has proved a q -analog of (4)

[6]. (Although his proof works only for $a = q^{-k}$, the formula remains true as a power series in t for any a .) The result is

$$\sum_{n=0}^{\infty} \frac{(abq)_n (aq)_n}{(q)_n (bq)_n} (bq)^n q^{(3)} P_n(x; a, b, q) P_n(y; a, b, q) t^n$$

$$= \frac{(-abqt)_{\infty}}{(-t)_{\infty}} \sum_{n,m=0}^{\infty} \frac{(abq)_{2(n+m)} (bqx)_m (bqy)_m (xbyq^2)^n q^m}{(q)_m (q)_n (aq)_n (bq)_m (-abqt)_{n+m} (-q/t)_{n+m}} \quad (5)$$

where $(a)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$.

A q -analogue of the Poisson kernel for Jacobi polynomials [2, equation (2.32)] is

$$\sum_{n=0}^{\infty} \frac{(abq)_n (aq)_n}{(q)_n (bq)_n} (1 - abq^{2n+1}) (bq)^n q^{(3)} P_n(x; a, b, q) P_n(y; a, b, q) t^n$$

$$= \frac{(-abq^3 t)_{\infty}}{(-t)_{\infty}} (1 - abq^2 t^2) (1 - abq)$$

$$\cdot \sum_{n,m=0}^{\infty} \frac{(abq^2)_{2(n+m)} (bqx)_m (bqy)_m (xybq^2)^n q^m}{(q)_m (q)_n (aq)_n (bq)_m (-abq^3 t)_{n+m} (-q/t)_{n+m}}.$$

This is not the Poisson kernel for q -Jacobi polynomials because there is no $q^{(3)}$ factor in the L^2 -norm [1, Theorem 9].

The same technique proves the Hille-Hardy formula for Laguerre polynomials and Mehler's formula for Hermite polynomials.

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