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CONCERNING DANILJUK'S EXISTENCE THEOREM FOR FREE BOUNDARY-VALUE PROBLEMS WITH THE BERNOULLI CONDITION

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ABSTRACT. We show that a variational method proposed by Ī. Ī. Daniljuk for proving the existence of free boundaries satisfying the Bernoulli condition is not valid under the conditions given by Daniljuk.

1. Introduction. Although variational techniques have often been used to obtain existence results for free boundary-value problems involving the Bernoulli condition (see **[8]**, **[9]** and **[10]**), apparently the most general result of this type in the literature was published by Ī. Ī. Daniljuk **[6]** in 1968. (The result was also stated in **[7]**.) In this note, we show that one significant aspect of Daniljuk's method is in general invalid, and we give a counterexample (in §4) which, we hope, sheds light on the ambiguous relationship between free boundary-value problems and their associated free-boundary extremal problems. Finally (in §5), we explain what Daniljuk has actually shown in **[6]** and compare his results with (mainly) the author's results for closely related free boundary problems.

2. The free boundary-value problem. Daniljuk considers the following problem: Given are a bounded, simply-connected region G in the plane $\mathbf{R}^2$, bounded by a simple closed curve Γ, and a strictly positive, continuous function $Q(p)$ defined on $G \cup \Gamma$. We seek a simple closed curve $\gamma \subset G$ such that if G_γ is the doubly-connected region bounded by $\Gamma \cup \gamma$ and the function $\psi_\gamma(p)$ solves the boundary-value problem

$$\nabla^2\psi_\gamma = 0 \text{ in } G_\gamma, \qquad \psi_\gamma = 0 \text{ on } \gamma, \qquad \psi_\gamma = 1 \text{ on } \Gamma, \tag{1}$$

then

$$|\nabla\psi_\gamma(p)| = Q(p) \quad \text{on } \gamma \tag{2}$$

(i.e., for each $p \in \gamma$, $\lim_{q\to p}|\nabla\psi_\gamma(q)| = Q(p)$, $q \in G_\gamma$).

3. Daniljuk's result. Let $\mathbf{X}$ denote the set of all simple closed curves $\gamma \subset G$, and define the functional $I(\gamma)$: $\mathbf{X} \to \mathbf{R}$ by

$$I(\gamma) = \int\int_{G_\gamma} \left(|\nabla\psi_\gamma(p)|^2 + Q^2(p)\right) dx\, dy > 0. \tag{3}$$

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In [6], Daniljuk claims to prove the following two assertions.

1. If Γ is rectifiable, and

$$d := \inf\{I(\gamma)\colon \gamma \in \mathbf{X}\} < \int\int_G Q^2(p)\,dx\,dy, \tag{4}$$

then there exists a curve $\gamma^* \in \mathbf{X}$ such that $I(\gamma^*) = d$.

2. If, in addition, Γ has Hölder continuous curvature and the function $Q(p) = Q(x,y)$ has Hölder continuous first derivatives, then γ^* is a Liapunov curve which satisfies (2), i.e., $|\nabla\psi_{\gamma^*}(p)| = Q(p)$ on γ^*.

4. A counterexample. Given constants $L > R > 0$, let $G_h = [-L, L] \times (-h, h) \cup B_R(p_0) \cup B_R(-p_0)$ for each $0 < h < R$, where $\pm p_0$ denotes the point $(\pm L, 0)$ (see Figure 1). For $0 < h < R$, let Γ_h be the boundary of G_h and let $\mathbf{X}_h$ be the set of all simple closed curves $\gamma \subset G_h$. Also, for $0 < h < R$, $c > 0$ and $\gamma \in \mathbf{X}_h$, let the notation $G_{h,\gamma}$, $\psi_{h,\gamma}(p)$ and $I_{h,c}(\gamma)$ correspond to G_γ, $\psi_\gamma(p)$ and $I(\gamma)$ in the case where $G = G_h$ and $Q(p) = c$ in $G_h \cup \Gamma_h$.

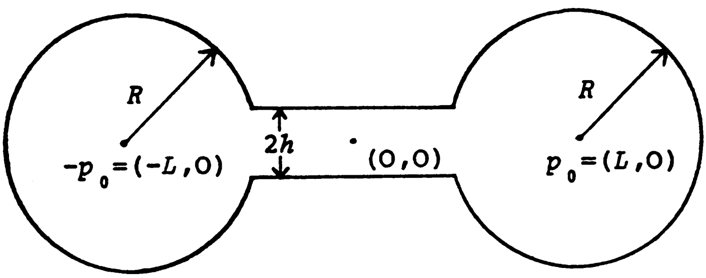

FIGURE 1. The region G_h

We will determine values h and c such that

$$\inf\{I_{h,c}(\gamma)\colon \gamma \in \mathbf{X}_h\} < \int\int_{G_h} c^2\,dx\,dy \tag{5}$$

(condition (4)), yet no curve $\gamma^* \in \mathbf{X}_h$ can simultaneously minimize $I_{h,c}(\gamma)$ in $\mathbf{X}_h$ and satisfy the Bernoulli condition $|\nabla\psi_{h,\gamma^*}(p)| = c$ on γ^*. For simplicity, our example does not take into account Daniljuk's assumption that Γ have Hölder continuous curvature. As will be seen, this assumption plays no essential role in the phenomenon to be exhibited. Our argument is divided into three steps.

Part 1. If $c > (\sqrt{2}\,e/R) + 1$ (where $e = \exp(1)$), then (5) holds for all sufficiently small $h > 0$.

PROOF. If $\gamma = \{p \in \mathbf{R}^2\colon |p - p_0| = r\}$ for some fixed value $0 < r < R$, then one easily sees that $\gamma \in \mathbf{X}_h$ for all $0 < h < R$, and

$$I_{h,c}(\gamma) = (2\pi/\log(R/r)) + c^2(\pi R^2 + \pi(R^2 - r^2)) + o(1) \quad \text{as } h \to 0+. \tag{6}$$

The assertion follows by setting $r = R/e$ in (6).

Part 2. There exists constants $0 < h_0 < R$ and $M > 0$ such that if $0 < h < h_0$ and $c^2h > M$, then any curve γ^* which minimizes $I_{h,c}(\gamma)$ in $\mathbf{X}_h$ must lie in $\mathbf{Y}_h :=$ the set of curves in $\mathbf{X}_h$ which intersect both $B_R(p_0)$ and $B_R(-p_0)$.

PROOF. Obviously, if $\gamma \in \mathbf{X}_h \setminus \mathbf{Y}_h$, then either $B_R(p_0) \subset G_\gamma$ or $B_R(-p_0) \subset G_\gamma$, and thus $I_{h,c}(\gamma) > \pi c^2 R^2$. Thus, it suffices to show (for $0 < h < h_0$ and $c^2 h > M$) that

$$I_{h,c}(\gamma_h) < \pi c^2 R^2 \tag{7}$$

for some specific curve $\gamma_h \in \mathbf{Y}_h$. For each $0 < h < R/2$, let $\gamma_h = \{p \in G_h\colon \text{distance}(p, \Gamma_h) = h/2\}$. One easily sees that

$$I_{h,c}(\gamma_h) = (4\pi R + 2(L - R))\left[(2/h) + b(h) + (hc^2/2)(1 + \varepsilon(h))\right], \tag{8}$$

for all $c > 0$, where $b(h)$ is bounded and $\varepsilon(h) \to 0$ as $h \to 0+$. The assertion follows by combining (7) and (8).

Part 3. If $h > 0$ is sufficiently small, then no curve $\gamma \in \mathbf{Y}_h$ can satisfy the Bernoulli condition

$$|\nabla \psi_{h,\gamma}(p)| = \sqrt{M/h} \quad \text{on } \gamma. \tag{9}$$

PROOF. For each $0 < h < R$ and $\gamma \in \mathbf{X}_h$, the function $u_{h,\gamma}(p) := \log(|\nabla \psi_{h,\gamma}(p)|)$ is harmonic in $G_{h,\gamma}$. Since $\psi_{h,\gamma}(p)$ can be harmonically continued across the analytic portions of Γ_h, we conclude, using the fact that $D_y^2\psi_{h,\gamma}(p) = -D_x^2\psi_{h,\gamma}(p) = 0$ on $\Lambda_h := [-L + R, L - R] \times \{h\} \subset \Gamma_h$ (where $D_y = \partial/\partial y$, etc.), that

$$D_y u_{h,\gamma}(p) = 0 \text{ in } \Lambda_h, \qquad 0 < h < R. \tag{10}$$

One can also show

$$\inf\{u_{h,\gamma}(p)\colon \gamma \in \mathbf{Y}_h, p \in \Lambda_h\} > \log(1/2h) + O(1) \quad \text{as } h \to 0+. \tag{11}$$

Since $u_{h,\gamma}(p) = \log(\sqrt{M/h}\,)$ for all $0 < h < R$, $\gamma \in \mathbf{X}_h$ satisfying (9), and $p \in \gamma$, one concludes that if we set $E_{h,\gamma} = \int_{\Lambda_h} D_y u_{h,\gamma}(p)\, dx$, then

$$\inf\{E_{h,\gamma}\colon \gamma \in \mathbf{Y}_h, \gamma \text{ satisfies (9)}\} > (L - R)\frac{\log(1/2h) - \log\left(\sqrt{M/h}\,\right)}{2h} + O(1) \tag{12}$$

as $h \to 0+$ (where $\inf \varnothing = \infty$). Since the right-hand side of (12) approaches infinity as $h \to 0+$, we conclude using (10) that the set on the left-hand side of (12) is empty for all sufficiently small $h > 0$.

5. Concluding discussion. In order to discuss Daniljuk's error in [**6**], we let $\tilde{\mathbf{X}}$ denote the set of all interior boundary components γ (other than points) of doubly-connected regions G_γ having Γ as their exterior boundary. For Γ rectifiable, Daniljuk's argument in [**6**, §5] correctly shows (under assumption (4)) that the functional $I(\gamma)\colon \tilde{\mathbf{X}} \to \mathbf{R}$ defined by (1) and (3) is minimized by some $\gamma^* \in \tilde{\mathbf{X}}$. However, he is incorrect in asserting that $\gamma^* \in \mathbf{X}$ and that γ^* is a Liapunov curve. Thus, the variational procedure in [**6**, §3], which he uses to show that γ^* satisfies (2), cannot be applied. In fact γ^* does not in general satisfy (2), as our argument in §4 shows. Possibly, Daniljuk's method can be made rigorous, under appropriate additional assumptions, by the addition of a proof that if γ^* minimizes $I(\gamma)$ in $\tilde{\mathbf{X}}$, then actually $\gamma^* \in \mathbf{X}$. (Daniljuk's proof that γ^* is a Liapunov curve would apparently then be correct.) However, in this author's opinion, this proof will probably turn out to be the most difficult aspect of the entire method. Nevertheless, rigorous existence results (under much stronger assumptions) for a related free boundary problem involving cavitational flow were obtained by essentialy the variational method proposed by Daniljuk in **[9]**. (See also **[8]** and **[10]**.)

REMARK 1. In **[3]**, the author has shown by a quite different variational method that if the set $\{(p, z) \in G \times \mathbf{R}: 0 < z < Q(p)\}$ is convex in $\mathbf{R}^3$, i.e., G is convex and the function $Q(p) > 0$ is concave in G, then for any value $0 < A < \|G\|$ (where $\|\cdot\|$ denotes area weighted by $Q^2(p)$), there exists a convex curve $\gamma^* \in \mathbf{X}$ such that $\|G_{\gamma^*}\| = A$ and $|\nabla\psi_{\gamma^*}(p)| = c \cdot Q(p)$ on γ^* for some constant $c > 0$. Under the same assumptions concerning G and $Q(p)$, it is also shown in **[3]** that if $I(\gamma) < \|G\|$ for some convex curve $\gamma \in \mathbf{X}$, then there exists a convex curve $\gamma^* \in \mathbf{X}$ such that $|\nabla\psi_{\gamma^*}(p)| = Q(p)$ on γ^*.

REMARK 2. In a slight variation of the free boundary-value problem in §2, one can let G denote the exterior complement of Γ, and redefine the remaining notation correspondingly. In **[1]**, the author showed that if $S := \mathbf{R}^2 \setminus G$ is starlike relative to a point $p_0 \in S$, and if $\lambda \cdot Q(p_0 + \lambda(p - p_0))$ is (weakly) monotone increasng in $\lambda > 1$ for each $p \in \Gamma$, then the functional $I(\gamma)$: $\mathbf{X} \to \mathbf{R}$ defined by (3) (where $\mathbf{X}$ is the set of simple closed curves which encircle S) is *uniquely* minimized by a curve $\gamma^* \in \mathbf{X}$, which satisfies the Bernoulli condition (2).

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