PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, November 1980

k-SPACES AND PRODUCTS OF CLOSED IMAGES OF METRIC SPACES

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ABSTRACT. We show that a recent theorem of Y. Tanaka giving necessary and sufficient conditions for the product of two closed images of metric spaces to be a k-space is independent of the usual axioms of set theory.

1. Introduction. According to Y. Tanaka [T_1], a space X is said to belong to *class* $\mathfrak{X}'$ if it is the union of countably many closed and locally compact subsets X_n such that $A \subset X$ is closed whenever $A \cap X_n$ is closed in X_n for all $n = 1, 2, \ldots$. In a later paper [T_3], Tanaka, assuming the continuum hypothesis (CH), gives necessary and sufficient conditions for the product of two closed images of metric spaces to be a k-space:

THEOREM (CH). *Let X and Y be closed images of metric spaces. Then $X \times Y$ is a k-space if and only if one of the following holds*:

(1) *X and Y are metric spaces*;

(2) *X or Y is a locally compact metric space*;

(3) *X and Y are in the class* $\mathfrak{X}'$.

Of course, it is natural to ask if the assumption of CH is necessary. In this paper, we shall show that this theorem is in fact equivalent to a certain set-theoretic axiom weaker than CH. At first glance, this may seem a bit odd. But it turns out that the truth of the theorem depends on its truth for a very special class of spaces. If σ is a cardinal number, let S_σ be the space obtained from the disjoint union of σ convergent sequences by identifying all the limit points to a single point. We will show that the truth of the theorem depends on whether or not $S_\omega \times S_{\omega_1}$ is a k-space. It is not so surprising that this depends on your set theory.

2. Main results. We use the following conventions. If A and B are sets, then ${}^AB$ is the set of all functions from A into B. Cardinals are initial ordinals, and an ordinal is the set of its predecessors.

For two functions f and g from ω to ω, we define $f < g$ if and only if the set $\{n \in \omega: f(n) > g(n)\}$ is finite. If κ is a cardinal number, then BF(κ) is the following assertion.

BF(κ): If $F \subset {}^\omega\omega$ has cardinality less than κ, then there exists $g \in {}^\omega\omega$ such that $f < g$ for all $f \in F$.

Received by the editors September 26, 1979; presented to the Society, January 25, 1980.
1980 *Mathematics Subject Classification*. Primary 54D50; Secondary 54B10, 54C10.
Key words and phrases. k-space, BF(κ).

It is known that Martin's Axiom implies that BF(κ) holds for all κ less than or equal to the continuum. It is easy to observe that CH implies that BF(ω_2) is false. Applying BF(κ) to topological problems is not new–see **[vDW]**, for example. The following lemma demonstrates why BF(κ) is of interest to us.

LEMMA 1. *Let κ be a cardinal number, and let κ^+ denote the least cardinal greater than κ. Then $S_\omega \times S_\kappa$ is a k-space (or a sequential space) if and only if* BF(κ^+).

PROOF. Suppose BF(κ^+) is false. Then there exists a collection $\{f_\alpha \in {}^\omega\omega: \alpha < \kappa\}$ such that if $f \in {}^\omega\omega$, then there exists $\alpha < \kappa$ with $f_\alpha(n) > f(n)$ for infinitely many $n \in \omega$.

For each $\alpha < \kappa$, let $H_\alpha = \{(m_n, n_\alpha) \in S_\omega \times S_\kappa: m \leqslant f_\alpha(n)\}$, where m_n and n_α denote the mth term of the nth sequence in S_ω and the nth term of the αth sequence in S_κ, respectively. Let $H = \bigcup_{\alpha<\kappa} H_\alpha$.

We will prove that H is k-closed, but not closed. If K is a compact set in $S_\omega \times S_\kappa$, then K meets only finitely many sequences in each factor. Thus K meets only finitely many H_α's. Since each H_α is closed, $K \cap H$ is closed. Thus H is k-closed. Now let ∞ denote the nonisolated point in S_ω and S_κ. Suppose U is any open set in $S_\omega \times S_\kappa$ containing (∞, ∞). Let $f \in {}^\omega\omega$ and $g \in {}^\kappa\omega$ be such that $(m_k, n_\alpha) \in U$ whenever $f(k) \leqslant m$ and $g(\alpha) \leqslant n$. There exists $\alpha < \kappa$ such that $f_\alpha(n) > f(n)$ for infinitely many $n \in \omega$. Thus there exists $n' \in \omega$ such that $g(\alpha) \leqslant n'$ and $f_\alpha(n') > f(n')$. This implies that $(f_\alpha(n')_{n'}, n'_\alpha) \in H_\alpha \cap U$. So we have shown that $(\infty, \infty) \in \overline{H} \setminus H$, and yet H is k-closed. Thus $S_\omega \times S_\kappa$ is not a k-space. This proves the "only if" part of Lemma 1.

Now assume BF(κ^+) holds. Suppose H is a subset of $S_\omega \times S_\kappa$ which is sequentially closed, but not closed. Then $(\{n_m\} \times S_\kappa) \cap H$ and $(S_\omega \times \{n_\alpha\}) \cap H$ are closed for each $n_m \in S_\omega$ and $n_\alpha \in S_\kappa$, so $\overline{H} = H \cup \{(\infty, \infty)\}$. Also, $H \cap [(\{\infty\} \times S_\kappa) \cup (S_\omega \times \{\infty\})]$ is closed (if not, (∞, ∞) would be a sequential limit point of H), so we may assume witout loss of generality that H contains only isolated points.

If $f \in {}^\omega\omega$, let U_f be the open set in S_ω defined by $U_f = \{\infty\} \cup \{m_n: m \geqslant f(n)\}$. For $g \in {}^\kappa\omega$, define analogously an open set V_g in S_κ.

Since $S_\omega \times (\{n_\alpha: n \in \omega\} \cup \{\infty\})$ is sequential for fixed α, there exists $f_\alpha \in {}^\omega\omega$ and $n(\alpha) \in \omega$ such that $(U_{f_\alpha} \times \{m_\alpha: m \geqslant n'(\alpha)\}) \cap H = \varnothing$. Applying BF($\kappa^+$), there exists a function f which bounds all the f_α's, $\alpha < \kappa$. For each α, let $k(\alpha) \in \omega$ be such that $f_\alpha(k) \leqslant f(k)$ whenever $k \geqslant k(\alpha)$.

For each $\alpha < \kappa$, there exists $n'(\alpha) \in \omega$ with $n'(\alpha) \geqslant n(\alpha)$ such that $((\bigcup_{j<k(\alpha)}\{m_j: m \in \omega\}) \times \{m'_\alpha: m' \geqslant n(\alpha)\}) \cap H = \varnothing$, for otherwise (x, ∞) would be a sequential limit point of H for some $x \in S_\omega$. Now define $g \in {}^\kappa\omega$ by $g(\alpha) = n'(\alpha)$. Suppose $(m_n, m'_\alpha) \in U_f \times V_g$. If $n < k(\alpha)$, then since $m' \geqslant n'(\alpha)$, it is true that $(m_n, m'_\alpha) \notin H$. If $n \geqslant k(\alpha)$, then $f_\alpha(n) \leqslant f(n) \leqslant m$, and so $(m_n, m'_\alpha) \in U_{f_\alpha} \times \{p_\alpha: p \geqslant n(\alpha)\}$, which implies $(m_n, m'_\alpha) \notin H$. Thus $(U_f \times V_g) \cap H = \varnothing$, contradicting the assumption that H was not closed. It follows that $S_\omega \times S_\kappa$ is sequential (and thus a k-space).

A space X is a *Fréchet space* if whenever $A \subset X$ and $x \in \bar{A}$, then there exists a sequence $x_1, x_2, \ldots$ contained in A which converges to x.

LEMMA 2. *Let f: $X \to Y$ be a closed map with X a collectionwise-normal Fréchet space. Let $y \in Y$. If $\partial f^{-1}(y)$ contains a closed discrete subset of cardinality σ, then Y contains a closed subset homeomorphic to S_σ.*

PROOF. Suppose D is a closed discrete subset of $f^{-1}(y)$ of cardinality σ. Let $\{U_d$: $d \in D\}$ be a discrete collection of open sets with $d \in U_d$. For each $d \in D$, let d_0, $d_1, d_2, \ldots$ be a sequence in $U_d \setminus f^{-1}(y)$ converging to d, such that $f(d_n) \neq f(d_{n'})$ whenever $n, n' \in \omega$, $n \neq n'$. For each $d \in D$ and $n \in \omega$, let $E(d_n) = \{d' \in D$: $f(d'_m) = f(d_n)$ for some $m \in \omega\}$. Since f is closed, it is easy to see that for each $d \in D$, there exists $n(d)$ such that $\bigcup_{n=n(d)}^{\infty} E(d_n)$ is finite for every $d \in D$.

Now pick $d(0) \in D$. If $d(\alpha)$ has been defined for all $\alpha < \beta$, where $\beta < \sigma$, pick $d(\beta) \in D \setminus \bigcup_{\alpha<\beta} E(d(\alpha))$. Let $D^* = \{d(\alpha)$: $\alpha < \sigma\} \cup \{d(\alpha)_n$: $\alpha < \sigma$ and $n \in \omega\}$. Then D^* is a closed subset of X, and f is one-to-one on $D^* \setminus f^{-1}(y)$. Thus $f(D^*)$ is homeomorphic to S_σ and is closed in Y.

Now we describe another simple space which will have a role to play later on. For each $n \in \omega$, let T_n be a countably infinite set. Let $T = (\bigcup_{n\in\omega} T_n) \cup \{\infty\}$, where the points of T_n are isolated for each $n \in \omega$, and a basic open set containing ∞ has the form $\{\infty\} \cup (\bigcup_{n>k} T_n)$, where $k \in \omega$.

LEMMA 3. *If X is a regular first countable space and X is not locally countably compact, then X contains a closed subset homeomorphic to T.*

PROOF. We omit the straightforward proof of this lemma.

LEMMA 4. *$S_\omega \times T$ is not a k-space.*

PROOF. Let $t_{n,m}$ denote the mth term in some enumeration of T_n. As before, let m_n denote the mth term of the nth sequence in S_ω. Let $H = \{(m_n, t_{n,m}) \in S \times T$: $n, m \in \omega\}$. It is easy to check that H is k-closed but not closed, and so $S \times T$ is not a k-space.

Now we are ready for the main result.

THEOREM 1. *The following are equivalent*:

(a) *$S_\omega \times S_{\omega_1}$ is not a k-space*;

(b) BF(ω_2) *is false*;

(c) *if X and Y are closed images of metric spaces, then $X \times Y$ is a k-space if and only if one of the following holds*:

(i) *X and Y are metrizable*;

(ii) *X or Y is locally compact and metrizable*;

(iii) *X and Y are in the class $\mathfrak{T}'$.*

PROOF. We have (a) $\leftrightarrow$ (b) from Lemma 1. It is easy to check that S_{ω_1} is not in class $\mathfrak{T}'$. So we also have (c) $\to$ (a). Assume (a) holds. Tanaka's proof of the "if" part of (c) does not use any axioms of set theory beyond ZFC, so we just need to prove the "only if" part. To this end, assume X and Y are closed images of metric spaces, and that $X \times Y$ is a k-space. If X and Y are closed s-images (i.e., point-inverses are separable) of metric spaces, then Tanaka [T_1] has shown that (i), (ii), or (iii) holds. So assume X is not a closed s-image of a metric space. Then by Lemma 2, X contains a closed subset homeomorphic to S_{ω_1}. Then from (a), and the assumption that $X \times Y$ is a k-space, we see that Y does not contain a closed subset homeomorphic to S_ω. But Y is the closed image of a metric space, and so by Lemma 2, point-inverses must have compact boundaries. Thus Y is metrizable. If Y were not locally compact, then by Lemmas 3 and 4, $X \times Y$ would not be a k-space. So Y is locally compact, and (ii) holds. This finishes the proof.

Now we consider a related question. In [T_2], Tanaka asks the following: Let Y be a closed image, under a map f, of a metric space. If Y^2 is a k-space, must each $\partial f^{-1}(y)$ be a Lindelöf space? Tanaka showed in [T_3] that the answer is yes, assuming CH. By Lemma 2, we see that if some $\partial f^{-1}(y)$ is not Lindelöf, then Y contains a closed copy of S_{ω_1}. So we could like to know if $S_{\omega_1}^2$ is a k-space. It turns out that this can be determined in ZFC.

LEMMA 5. *$S_{\omega_1}^2$ is not a k-space..*

PROOF. For each $\alpha \in \omega_1$, let $f_\alpha\colon \omega_1 \to \omega$ be a function such that f_α restricted to α is a one-to-one map onto ω. Define $H_\alpha = \{(m_\beta, f_\alpha(\beta)_\alpha) \in S_{\omega_1}^2 \colon m < f_\alpha(\beta)\}$, and let $H = \bigcup_{\alpha < \omega_1} H_\alpha$.

We claim that H is k-closed, but not closed in $S_{\omega_1}^2$. Suppose K is a compact subset of $S_{\omega_1}^2$. Then K meets only finitely many sequences in each factor, and thus only finitely many H_α's. Let $\beta(1), \beta(2), \ldots, \beta(n)$ be the sequences in the first factor that K meets. Then $K \cap H_\alpha = \{(m_{\beta(i)}, f_\alpha(\beta(i))_\alpha) \in S_{\omega_1}^2 \colon m < f_\alpha(\beta(i)), i = 1, 2, \ldots, n\}$, which is a finite set. Thus $K \cap H$ is finite and hence closed.

It remains to prove that H is not closed. Let $g\colon \omega_1 \to \omega$, and let U_g be the open set in S_{ω_1} containing (∞, ∞) determined by g, as defined in the proof of Lemma 1. There exists $n_0 \in \omega$ and an uncountable subset A of ω_1, such that $g(\alpha) = n_0$ whenever $\alpha \in A$. Let γ be an element of A which has infinitely many predecessors in A. There exists $\delta \in A$ with $\delta < \gamma$ and $f_\gamma(\delta) = m > n_0$. Then $(m_\delta, m_\gamma) \in H_\gamma \cap (U_g \times U_g)$. We have shown, then, that (∞, ∞) must be a limit point of H. Therefore, H is not closed and $S_{\omega_1}^2$ is not a k-space.

The following theorem answers the question of Tanaka referred to above.

THEOREM 2. *Let Y be a closed image, under a map f, of a paracompact Fréchet space. If Y^2 is a k-space, then for each $y \in Y$, $\partial f^{-1}(y)$ is Lindelöf.*

PROOF. Let Y satisfy the hypotheses. If some $\partial f^{-1}(y)$ is not Lindelöf, then by Lemma 2, Y contains a closed copy of S_{ω_1}. But $S_{\omega_1}^2$ is not a k-space, so in this case Y^2 is not a k-space.

The next theorem is the same as [T_2, Proposition 2.7], but we do not need the assumption of CH.

THEOREM 3. *Let X be a locally compact, paracompact, first countable space, and let A be a closed subset of X. Then $(X/A)^2$ is a k-space if and only if ∂A is Lindelöf.*

PROOF. If $(X/A)^2$ is a k-space, then ∂A is Lindelöf by Theorem 2. On the other hand, if ∂A is Lindelöf, then Tanaka's proof that $(X/A)^2$ is a k-space can be used, since this part of his proof does not depend on CH (or anything else beyond ZFC).

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