HOMOTOPY NILPOTENCY AND ASSOCIATIVITY

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ABSTRACT. If (Y, μ) is an H -space then a measure of nonassociativity, called Homotopy Nuclear Nilpotency (HNN) is defined. We then calculate the HNN of thirteen H -spaces: the twelve classes of multiplications on S^3 and the one class on $S^5_{(2)}$, the 5-sphere localized away from 2.

1. If (Y, μ) is a homotopy associative H -space, Bernstein and Ganea [B-G] defined Y to have homotopy nilpotency less than n if, for every space X , the group $[X, Y]$ is a nilpotent group of class less than n . It is easily seen that this is equivalent to the statement that $\beta_n: Y^n \rightarrow Y$ is homotopic to a constant map, where $\beta_n = \beta_2(\beta_{n-1} \times 1)$, $\beta_2(x, y) = x^{-1}y^{-1}xy$ and x^{-1} denotes the unique (up to homotopy) inverse of x . This second definition was used by Arkowitz and Curjel [A-C] and Gilbert [G] to extend the definition and calculations of homotopy nilpotency to the nonassociative multiplications on the H -spaces S^3 and S^7 where the homotopy inverse is well defined. Finally Zabrodsky [Z] defines homotopy nilpotency for an arbitrary H -space essentially by letting β_2 be such that $\mu(\beta_2 \times \mu T)\Delta \sim \mu$. (Here T is the twist map and Δ the diagonal.) In each of these cases the homotopy nilpotency of Y gives some measure of the lack of commutativity of the multiplication μ . However, if the multiplication μ on Y is not associative, $[X, Y]$ forms an algebraic loop (not necessarily associative group) and for algebraic loops Bruck [B] has defined the concept of nuclear nilpotency as a measure of the nonassociativity of the loop. This definition may be worded as follows.

Let G be a loop and N a normal subloop (kernel). Define $[G, N]$ to be the normal subloop of G generated by all x in G which satisfy one of the equations in (1.1).

CONDITION (1.1).

$$(cx)(ab) = (ca)b, a((cx)b) = (ac)b, (ab)(cx) = a(bc) \quad \text{for } a, b \text{ in } G, c \text{ in } N.$$

Define $\Gamma_1 = G$ and $\Gamma_n = [G, \Gamma_{n-1}]$. Then G is *nuclear nilpotent of class r* if $\Gamma_{r+1} = 1, \Gamma_r \neq 1$.

With this in mind we define the concept of Homotopy Nuclear Nilpotency.

DEFINITION 1.2. An H -space (Y, μ) is *homotopy nuclear nilpotent of class less than n* (HNN(Y) $< n$) if for all spaces X the loop $[X, Y]$ is nuclear nilpotent of class less than n . Thus HNN(Y) < 2 is equivalent to the statement that Y is homotopy associative.

In §2 we will prove the following theorem.

Received by the editors October 18, 1979.

1980 *Mathematics Subject Classification*. Primary 55P45.

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0002-9939/80/0000-0573/\$02.00

THEOREM 1.3. *For every multiplication on S^3 , $HNN(S^3) < 3$.*

Note that of the 12 nonhomotopic multiplications on S^3 , 8 are homotopy associative and for these the theorem is trivial. However, for the remaining four multiplications, Theorem 1.3 insures that they are as close to associative as possible. In §3 we will calculate the HNN of $X = S^5_{(2)}$ (the 5-sphere S^5 localized away from 2).

If we replace the equations (1.1) with the equation $x(ac) = ca$ we can get a loop-theoretic definition of nilpotency which appears to correspond to Zabrodsky's topological definition. However while it is easy to show that $\beta_n \sim *$ implies the algebraic definition, the converse is an open question. Similarly there is, at present, no equivalent topological definition of homotopy nuclear nilpotency.

2. Nuclear nilpotency of S^3 . Let (S^3, μ) be the H -space S^3 with the standard quaternionic multiplication. For any space W let $[W, S^3]$ be the group of based homotopy classes of maps together with a product induced by this multiplication. Both of these products will be indicated by juxtaposition. Any of the 12 nonhomotopic multiplications μ_r , $r = 0, 1, \dots, 11$, induce a loop structure $[W, S^3]_r$ with the multiplication $\circ$ defined by $a \circ b = ab[a, b]_r$ where $[a, b]_r = a^{-1}b^{-1}ab$. By [N, 9.3] each of these loops is a Moufang loop (i.e., $(a \circ b) \circ (c \circ a) = (a \circ (b \circ c)) \circ a$) and hence is diassociative (the subloop generated by any two elements is a group). Also note that $a \circ a^{-1} = 1 = a^{-1} \circ a$.

Let $a, b \in [W, S^3]_r$ and define the commutator by the formula $[a, b]_r = a^{-1} \circ b^{-1} \circ a \circ b$. Diassociativity allows us to omit parentheses from this expression.

LEMMA 2.1. $[a, b]_r$ is in the center of the subgroup generated by a and b .

PROOF. The elements a, b induce an element (a, b) of $[W, S^3 \times S^3]$. Then the image $(a, b)*([S^3 \times S^3, S^3]_r)$ is a subloop of $[W, S^3]_r$ containing a, b . Further $(a, b)*([p_1, p_2]_r, p_1)_r = [[a, b]_r, a]_r$. But by [N, 5.8] $[[p_1, p_2]_r, p_1]_r = 1$. This implies the lemma by [B, VII, 5.1].

The following two corollaries are immediate.

COROLLARY 2.2. $[a, b]_r = [a^{-1}, b^{-1}]_r$, $[a^{-1}, b]_r = [a, b]_r^{-1}$.

COROLLARY 2.3. [A-C, 4]. $[a, b]_r = [a, b]^{2r+1}$.

For a, b, c in $[W, S^3]_r$ the associator $[a, b, c]_r$ is defined by $(a \circ b) \circ c = (a \circ (b \circ c)) \circ [a, b, c]_r$. In a manner similar to Lemma 2.1 we can prove

LEMMA 2.4. (a) $[[a, b, c]_r, a]_r = 1$; (b) $[[a, c]_r, [b, c]_r]_r = 1$.

PROPOSITION 2.5. Let $a, b, c \in [W, S^3]_r$. Then $[a, b, c]_r^3 = 1$.

We will give two proofs of Proposition 2.5, a completely algebraic one and a shorter topological one. The advantage of the algebraic proof is that it does not depend on the homotopy groups of S^3 and as such can, and will, be used for other H -spaces.

PROOF. It $r \not\equiv 1 \pmod 3$ then $[W, S^3]_r$ is a group by [J1, Theorem (3.2)] and the result trivially holds.

If $r \equiv 1 \pmod 3$ then [A-C], [B] show that $[[a, b]_r, c]_r = 1$. This combined with Lemma 2.4(a) and [B, VII, 5.5] imply

$$[a \circ b, c]_r = [a, c]_r \circ [b, c]_r \circ [a, b, c]_r^3. \quad (2.6)$$

Using Corollary 2.3 and the definition of $\circ$ we get

$$[ab[a, b]_r^r, c]^{2r+1} = [a, c]^{2r+1} [b, c]^{2r+1} [[a, c]^{2r+1}, [b, c]^{2r+1}]^r \circ [a, b, c]_r^3.$$

By Lemma 2.4(b) we get

$$[ab[a, b]_r^r, c]^{2r+1} = [a, c]^{2r+1} [b, c]^{2r+1} \circ [a, b, c]_r^3. \quad (2.7)$$

But in any group $[[xy, z]] = [x, z][[x, z], y][y, z]$, so that

$$[ab[a, b]_r^r, c] = [ab, c][[ab, c], [a, b]_r^r][[a, b]_r^r, c] = [ab, c][[a, b], c]_r^r,$$

since $[[ab, c], [a, b]_r^r] = 1$ and [B, VII, 5.1]. But $[ab, c] = [a, c][[b, c][[a, c], b]]$, since $[[a, c], b]$ is in the center of $[W, S^3]$.

Thus

$$[ab[a, b]_r^r, c] = [a, c][b, c][[a, c], [[a, b], c]_r^r] = [a, c][b, c][[a, b], c]_r^{r-1},$$

since $[[a, c], b] = [[a, b], c]^{-1}$ [B, VII, 5.3]. Further [B, VII, 5.3] $[[a, b], c]^3 = 1$. But $r \equiv 1 \pmod 3$ so that

$$\begin{aligned} [ab[a, b]_r^r, c]^{2r+1} &= ([a, c][b, c])^{2r+1} \\ &= [a, c]^{2r+1} [b, c]^{2r+1} [[a, c], [b, c]]^{-(2r+1)r} \\ &= [a, c]^{2r+1} [b, c]^{2r+1} \quad \text{by [B, VII, 5.1].} \end{aligned}$$

Thus (2.7) becomes

$$[a, c]^{2r+1} [b, c]^{2r+1} = [a, c]^{2r+1} [b, c]^{2r+1} \circ [a, b, c]_r^3 \quad \text{or} \quad [a, b, c]_r^3 = 1.$$

Proposition 2.5 may be proven topologically by noting that the map $[a, b, c]_r$ factors as

$$W \xrightarrow{\Delta} W^3 \rightarrow \Lambda^3 W \rightarrow \Lambda^3 S^3 = S^9 \rightarrow S^3$$

and that $\pi_9(S^3) = \mathbb{Z}/3$.

LEMMA 2.8. $[[a, b, c]_r, d, e]_r^2 = 1$.

PROOF. First note that $[[a, b, c]_r, d, e]_r = 1$ if any of the entries are 1 so that $[[a, b, c]_r, d, e]_r$ factors as

$$W \xrightarrow{\Delta} W^5 \rightarrow \Lambda^5 W \rightarrow \Lambda^5(S^3) = S^{15} \rightarrow S^3.$$

But $\pi_{15}(S^3) = (\mathbb{Z}/2)^2$.

THEOREM 2.9. $[W, S^3]_r$ is nuclear nilpotential of class less than or equal to 2.

PROOF. By the general techniques of [B] we need only show $[[a, b, c]_r, d, e]_r = 1$.

However

$$[[a, b, c]_r, d, e]_r^3 = [[a, b, c]_r^3, d, e]_r = 1$$

by [B, VII, 5.22] and Proposition 2.5.

This combined with Lemma 2.8 completes the proof of 1.3.

3. Nuclear nilpotency of $S_{(2)}^5$. Let $X = S_{(2)}^5$ be the localization of S^5 away from the prime 2. Then X is an H -space and since $\pi_{10}(X) = 0$ ($\pi_{10}(S^5) = \mathbb{Z}/2$), we note that X has a unique (up to homotopy) H -structure and the multiplication is homotopy commutative.

PROPOSITION 3.1. *X is not homotopy associative.*

PROOF. If X were homotopy associative then X would have a projective 3-space, $XP(3)$, so that $H^*(XP(3); \mathbb{Z}/3) = \mathbb{Z}/3[x]/x^4$ with dimension of x equaling 6. But by Theorem 2 of [T], this is impossible.

PROPOSITION 3.2. *Let $a, b, c \in [W, X]$, then $[a, b, c]^3 = 1$. This is an immediate consequence of Bruck's Theorem [B, VII, 5.5].*

THEOREM 3.3. *$[W, X]$ is nuclear nilpotent of class 2.*

PROOF. As before we need only show that $[[a_1, a_2, a_3], a_4, a_5] = 1$. Furthermore it is sufficient to demonstrate this when $W = (S^5)^5$ and $a_i: W \rightarrow S^5$ is the projection in the i th component. But this map factors through $\Lambda^5(S^5) \rightarrow X$ and is the mapping $\langle\langle i, i, i \rangle, i, i \rangle$ defined by James [J2]. By Proposition 3.2 $\langle i, i, i \rangle = 3\alpha$ for some $\alpha \in \pi_{15}(X) = \mathbb{Z}/9$ so that $\langle\langle i, i, i \rangle, i, i \rangle = \langle 3\alpha, i, i \rangle = 3\langle \alpha, i, i \rangle$. But by Mimura [M-T] $\pi_{25}(X) = \mathbb{Z}/3$ ($\pi_{25}(S^5) = \mathbb{Z}/6 \times \mathbb{Z}/2 \times \mathbb{Z}/2$) and hence $\langle\langle i, i, i \rangle, i, i \rangle = 0$. This shows $NNP \leq 2$. Proposition 3.1 guarantees equality.

Observe that this proof imitates the algebraic proof of Proposition 2.5. The topological proof would be useless here, since it would involve factoring through $\pi_{15}(X) = \mathbb{Z}/9$ ($\pi_{15}(S^3) = \mathbb{Z}/72 + \mathbb{Z}/2$).

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