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AUTOMORPHISMS OF FIBRATIONS[1]

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ABSTRACT. Let X be a simplicial set, G a simplicial group and $\overline{W}G$ the classifying complex of G. Then it is well known [1], [3] that *the principal fibrations with base X and group G are classified by the components of the function* complex $(\overline{W}G)^X$. The aim of the present note is to prove the following complement to this result (1.2): *Let p be a principal fibration with base X and group G, and let* $\operatorname{aut} p$ *be its simplicial group of automorphisms* (which keep the base fixed). *Then* $\overline{W}(\operatorname{aut} p)$ *has the homotopy type of the component of* $(\overline{W}G)^X$ *which* (see above) *corresponds to p*. A similar result holds for ordinary fibrations.

1. Statements of results.

1.1. PRINCIPAL FIBRATIONS. Let X be a simplicial set, G a simplicial group and p: $E \to X$ a *principal fibration with group* G, i.e. [3, 18.1] G acts principally (i.e. freely) on E from the right, $X = E/G$ and p is the projection. Another principal fibration p': $E' \to X$ with group G is said to be *equivalent* to p if there exists a commutative diagram

$$\begin{array}{ccc} E & \rightarrow & E' \\ & p \searrow \quad \swarrow p' & \\ & X & \end{array}$$

in which the top map is compatible with the action of G (and hence an isomorphism). Denote by $\operatorname{aut} p$ the simplicial group of *automorphisms* of p, i.e. the simplicial group which has as n-simplices the commutative diagrams

$$\begin{array}{ccc} \Delta[n] \times E & \rightarrow & \Delta[n] \times E \\ & \mathrm{id} \times p \searrow \quad \swarrow \mathrm{id} \times p & \\ & \Delta[n] \times X & \end{array}$$

in which the top map is compatible with the action of G (and hence an isomorphism). Then one has

1.2. THEOREM. (i) *Let X be a simplicial set and G a simplicial group. Then the equivalence classes of principal fibrations with base X and group G are in* 1-1 *correspondence with the components of the function complex* $(\overline{W}G)^X$.

(ii) *Let p: $E \to X$ be a principal fibration with group G. Then* $\overline{W}(\operatorname{aut} p)$ *has the homotopy type of the component of* $(\overline{W}G)^X$ *which* (*see* (i)) *corresponds to p*.

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A similar result holds for ordinary

1.3. FIBRATIONS. Let X be a simplicial set, let M be a *minimal* [3, §9] simplicial set and let q: $Y \to X$ be a *fibration in which all the fibres have the homotopy type of* M. Another such fibration q': $Y' \to X$ is said to be *homotopy equivalent* to q if there exists a commutative diagram

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "q"'] & & Y' \arrow[dl, "q'"] \\ & X & \end{tikzcd}$$

in which the top map is a homotopy equivalence. Denote by haut q the simplicial monoid of *self-homotopy equivalences* of q, i.e. the simplicial monoid which has as n-simplices the commutative diagram

$$\begin{tikzcd} \Delta[n] \times Y \arrow[rr] \arrow[dr, "\mathrm{id} \times q"'] & & \Delta[n] \times Y \arrow[dl, "\mathrm{id} \times q"] \\ & \Delta[n] \times X & \end{tikzcd}$$

in which the top map is a homotopy equivalence and denote by aut M the simplicial group of *automorphisms* of M, i.e. the simplicial group which has as n-simplices the commutative diagrams

$$\begin{tikzcd} \Delta[n] \times M \arrow[rr] \arrow[dr, "\mathrm{proj}"'] & & \Delta[n] \times M \arrow[dl, "\mathrm{proj}"] \\ & \Delta[n] & \end{tikzcd}$$

in which the top map is an isomorphism. Then one has

1.4. THEOREM. (i) *Let X be a simplicial set and M a minimal simplicial set. Then the homotopy equivalence classes of fibrations with base X and all fibers homotopically equivalent to M are in* 1-1 *correspondence with the components of the function complex* $(\overline{W}(\mathrm{aut}\ M))^X$.

(ii) *Let q: $Y \to X$ be a fibration with all fibres homotopically equivalent to M. Then* $\overline{W}(\mathrm{haut}\ q)$ *has the (weak) homotopy type of the component of* $(\overline{W}(\mathrm{aut}\ M))^X$ *which (see* (i)) *corresponds to q.*

A further application, to *equivariant maps which are self-homotopy equivalences*, will be discussed in [2].

2. Proofs. In this section we make extensive use of [3], especially Chapter IV.

2.1. PROOF OF THEOREM 1.2. A proof of (i) is contained in [1] and [3, Chapter IV]. Given a principal fibration p: $E \to X$ with group G, it produces a map f: $X \to \overline{W}G$ as follows. Choose a *pseudo cross section* σ: $X \to E$ (i.e. a function such that $p\sigma = \mathrm{id}$, $d_i\sigma = \sigma d_i$ for $i > 0$ and $s_i\sigma = \sigma s_i$ for $i \geqslant 0$). This determines a *twisting function* τ: $X \to G$, which lowers dimensions by one and is given by $d_0\sigma x = (\sigma d_0 x)(\tau x)$ for all $x \in X$. The desired *classifying map* f: $X \to \overline{W}G$ then is defined by

$$fx = [\tau x, \tau d_0 x, \ldots, \tau d_0^{r-1} x] \quad \text{where } r = \dim x.$$

To prove (ii) consider the *universal* principal fibration sequence $G \to WG \to \overline{W}G$ and note that G *acts on this sequence from the left by conjugation*, i.e.

$$h\cdot[g_n,\ldots,g_0]=[hg_nh^{-1},(d_0h)g_{n-1}(d_0h)^{-1},\ldots,(d_0^nh)g_0(d_0^nh)^{-1}],$$
$$h\cdot[g_{n-1},\ldots,g_0]=[(d_0h)g_{n-1}(d_0h)^{-1},\ldots,(d_0^nh)g_0(d_0^nh)^{-1}].$$

Hence one can form the commutative diagrams

$$\begin{array}{ccccc}
G & \to & G\times_G WG & \to & \overline{W}G\\
\downarrow & & \downarrow & & \downarrow =\\
WG & \to & WG\times_G WG & \xrightarrow{\sim} & \overline{W}G\\
\downarrow & & \downarrow & & \downarrow =\\
\overline{W}G & \to & \overline{W}G\times_G WG & \to & \overline{W}G
\end{array}
\qquad
\begin{array}{ccccc}
G^X & \to & (G\times_G WG)^X & \to & (\overline{W}G)^X\\
\downarrow & & \downarrow & & \downarrow =\\
(WG)^X & \to & (WG\times_G WG)^X & \xrightarrow{\sim} & (\overline{W}G)^X\\
\downarrow & & \downarrow & & \downarrow =\\
(\overline{W}G)^X & \to & (\overline{W}G\times_G WG)^X & \to & (\overline{W}G)^X
\end{array}$$

in which, in the "products over G', G acts from the right on the simplicial sets on the right and from the left on the ones on the left. Clearly *the horizontal sequences are all fibration sequences with cross sections* and *the indicated* ($\sim$) *maps are homotopy equivalences*. Furthermore, if in the second diagram one considers the fibers G_f^X, $(WG)_f^X$ and $(\overline{W}G)_f^X$ over (see prior material) the vertex $f\in(\overline{W}G)^X$, then it is not hard to see that $(WG)_f^X$ *is contractible*, that G_f^X *is a simplicial group which acts principally on* $(WG)_f^X$ *from the right*, and that *the map* $(WG)_f^X/G_f^X\to(\overline{W}G)_f^X$ *is* 1-1 *and onto a single component of* $(\overline{W}G)_f^X$. Hence it remains to prove that

(I) G_f^X *is isomorphic* (*as a simplicial group*) *to* $\operatorname{aut}p$ and

(II) $(WG)_f^X/G_f^X$ *is isomorphic to the component of* $(\overline{W}G)^X$ *which contains* f.

Given an n-simplex $b\in\operatorname{aut}p$ (i.e. a right G-map $b\colon\Delta[n]\times E\to\Delta[n]\times E$ over $\Delta[n]\times B$), let $vb\colon\Delta[n]\times X\to G\times_G WG$ be the function given by

$$(a,x)\mapsto(b_ax,[*,\tau x,\tau d_0x,\ldots,\tau d_0^{r-1}x])$$

where $r=\dim(a,x)$ and $b_ax\in G$ is determined by (see preceding) the formula $(a,\sigma x)(b_ax)=b(a,\sigma x)$. Then a lengthy but straightforward calculation yields that vb is a simplicial map and that the resulting function $v\colon\operatorname{aut}p\to G_f^X$ is the desired isomorphism of simplicial groups.

Finally to prove (II) one notes that *the map* $\overline{W}G\times_G WG\to\overline{W}G\times\overline{W}G$ *given by*

$$([g_{n-1},\ldots,g_0],[*,h_{n-1},\ldots,h_0])\mapsto([k_{n-1},\ldots,k_0],[h_{n-1},\ldots,h_0]),$$
$$k_i=h_i(d_0h_{i+1})\cdots(d_0^{n-i-1}h_{n-1})g_i(d_0^{n-i-1}h_{n-1}^{-1})\cdots(d_0h_{i+1}^{-1}),\qquad 0<i<n,$$

is an isomorphism of simplicial sets and hence induces an isomorphism $(\overline{W}G)_f^X\approx(\overline{W}G)^X$, and it thus remains to show that, under this isomorphism, the image of $(WG)_f^X$ in $(\overline{W}G)_f^X$ goes to the component of $(\overline{W}G)^X$ containing f. But this readily follows from the fact that the image of $(WG)_f^X$ in $(\overline{W}G)_f^X$ contains the "vertex" $X\to\overline{W}G\times_G WG$ given by

$$x\mapsto([*,\ldots,*],[*,\tau x,\tau d_0x,\ldots,\tau d_0^{r-1}x])\quad\text{where } r=\dim x.$$

2.2. Proof of Theorem 1.4. Part (i) of Theorem 1.4 was proved in [1] and [3, Chapter IV] by reducing it to part (i) of Theorem 1.2. In the same manner part (ii) of Theorem 1.4 can be reduced to part (ii) of Theorem 1.2. One uses the following result, which is not hard to verify.

Let $q'\colon Y' \to X$ *be a minimal subfibration* [3, 10.17] *of* $q\colon Y \to X$ *and let* $p\colon E \to X$ *be the associated* [3, §§11, 19, 20] *principal fibration with* aut M *as group. Then*

(i) $\overline{W}$(haut q') *has the* (*weak*) *homotopy type of* $\overline{W}$(haut q),

(ii) haut q' *is actually a simplicial group, and*

(iii) haut q' *is isomorphic* (*as a simplicial group*) *to* aut p.

We end with an

2.3. Outline of a somewhat more conceptual proof of 1.2(ii). Let $(\overline{W}G)_p^X$ be the component of $(\overline{W}G)^X$ which corresponds (Theorem 1.2(i)) to p_X. Then one can construct a homotopy equivalence $\overline{W}(\operatorname{aut} p) \to (\overline{W}G)_p^X$ as follows.

The simplicial group aut p acts from the left on E and from the right on $W(\operatorname{aut} p)$. The resulting principal fibration $E \times_{\operatorname{aut} p} W(\operatorname{aut} p) \to X \times \overline{W}(\operatorname{aut} p)$ with group G has as pseudo cross section the function

$$\left(x, \left[a_{n-1}, \ldots, a_0\right]\right) \mapsto \left(\sigma x, \left[*, a_{n-1}, \ldots, a_0\right]\right)$$

where σ is the pseudo cross section of p of 2.1, and the desired map then is the adjoint of the resulting classifying map (see 2.1) $X \times \overline{W}(\operatorname{aut} p) \to \overline{W}G$.

To prove that this map $\overline{W}(\operatorname{aut} p) \to (\overline{W}G)_p^X$ is a homotopy equivalence one notes that, for every simplicial set Y, the homotopy classes of maps $Y \to \overline{W}(\operatorname{aut} p)$ classify principal fibrations with base $Y \times X$ and group G, subject to certain restrictions, and that the homotopy classes of maps $Y \to (\overline{W}G)_p^X$ do the same.

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