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MARTIN'S AXIOM IMPLIES THAT DE CAUX'S SPACE IS COUNTABLY METACOMPACT

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ABSTRACT. De Caux defined a space $S(\mathfrak{L})$ and, assuming ♣, showed that $S(\mathfrak{L})$ is normal but not countably metacompact. We assume MA_{ω_1} and show that $S(\mathfrak{L})$ is countably metacompact.

In [dC] Peter de Caux constructed a Dowker space assuming ♣. (For a definition of ♣ and review of the Dowker space problem, see [R].) Peter Nyikos [N] asked whether it was consistent with MA + ¬CH that de Caux's space be a Dowker space. Here we answer negatively by proving the title. (Martin's Axiom in the title means MA_{ω_1}.)

DEFINITION. A ladder system, $\mathfrak{L}$, is a sequence $(L_\lambda: \lambda \in \omega_1 \cap \mathrm{LIM})$ such that each L_λ is a cofinal subset of λ with order type ω.

DEFINITION. Given a ladder system $\mathfrak{L}$, we define a space $S(\mathfrak{L})$ as follows. The point set of $S(\mathfrak{L})$ is $\omega \times \omega_1$. We define a topology on $\omega \times \omega_1$ by defining a weak base $\mathfrak{B}_x$ at each point x. A set U will be open iff $x \in U$ implies there is $B \in \mathfrak{B}_x$, $B \subset U$. If x is of the form $(0, \alpha)$ or $(n, \alpha + 1)$, the only element of $\mathfrak{B}_x$ is $\{x\}$. If x is of the form $(n + 1, \lambda)$, where $\lambda \in \mathrm{LIM}$, then $\mathfrak{B}_x$ is the countable family of sets of the form $\{(n + 1, \lambda)\} \cup \{\{n\} \times (S_\lambda - e)\}$, where e is a finite set.

De Caux assumed ♣ to get the existence of a ladder system $\mathfrak{L}$ with special properties, and used the special properties to show that $S(\mathfrak{L})$ is a Dowker space. Nyikos calls any space of the form $S(\mathfrak{L})$ a de Caux's Litmus strip space. Clearly, one can construct $S(\mathfrak{L})$ without assuming ♣. Further, the properties of $S(\mathfrak{L})$ could vary with $\mathfrak{L}$. It is not hard to define $\mathfrak{L}$ so that $S(\mathfrak{L})$ is a σ-discrete Moore space, for example. What we are asserting in this paper is that assuming MA + ¬CH, all spaces of the form $S(\mathfrak{L})$ are countably metacompact.

DEFINITION. A space is countably metacompact iff whenever $(H_n)_{n\in\omega}$ is a decreasing sequence of closed sets with empty intersection then there is a sequence $(U_n)_{n\in\omega}$ of open sets with empty intersection satisfying $U_n \supset H_n$.

DEFINITION. A poset P has property K iff whenever W is an uncountable subset of P there is an uncountable subset W' of W of compatible elements. The product of property K posets has property K, a fortiori, ccc.

DEFINITION. For A a set $[A]^2$ is the set of two element subsets of A. If A is totally ordered by $<$, we may think of $[A]^2$ as $\{(a, b) \in A \times A: a < b\}$. A theorem of

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Dushnik and Miller[2] asserts that wherever $(A, <)$ has order type ω_1 and f: $[A]^2 \to \{0, 1\}$ then either there is a subset, S_0, of A of order type ω_1 such that $f(a, b) = 0$ for all $(a, b) \in [S_0]^2$ [we will abbreviate $f((a, b))$ by $f(a, b)$] or there is a subset, S_1, of order type $\omega + 1$ such that $f(a, b) = 1$ for all $(a, b) \in [S_1]^2$. We abbreviate this theorem by $\omega_1 \to (\omega_1, \omega + 1)^2_2$.

Now, assume we are given a space $S(\mathfrak{L})$, and a decreasing sequence $(H_n)_{n\in\omega}$ of closed sets with empty intersection. We will use MA + $\neg$CH to define a sequence $(U_n)_{n\in\omega}$ of open sets with empty intersection satisfying $U_n \supset H_n$.

We define P_n, a poset of approximations to U_n. Our plan is to define

$$U_n = \bigcup \{c_p : p \in G\} = \bigcup \left\{ \bigcup \text{ range } b_p : p \in G \right\}.$$

Let P_n be the set of triples $p = (a_p, c_p, b_p)$, usually abbreviated (a, c, b), satisfying

a is a finite subset of $S(\mathfrak{L}) - H_n$,
c is a finite subset of $S(\mathfrak{L})$,
b is a function with domain c, $b(x) \in \mathfrak{B}_x$,
$a \cap \bigcup$ range $b = \varnothing$.

We claim that P has property K. Let W be an uncountable subset of P. First we apply the Δ-system lemma to $\{a_p : p \in W\}$ to get an uncountable $W_1 \subset W$ such that $\{a_p : p \in W_1\}$ is a Δ-system with root r_a. Similarly obtain an uncountable $W_2 \subset W_1$ such that $\{c_p : p \in W_2\}$ is a Δ-system with root r_c. Next, find an uncountable $W_3 \subset W_2$ such that for all $p, q \in W_3$, $b_p | r_c = b_q | r_c$ and card $c_p = k =$ card c_q. Now we define $W_4 = \{p(\beta): \beta < \omega_1\}$ by induction on β so that the unfixed part of $p(\beta)$ is strictly above the sup of the unfixed parts of $p(\beta')$, $\beta' < \beta$. Precisely, for $x = (n, \alpha) \in S(\mathfrak{L})$ define $h(x) = \alpha$; define $h(a) = \max\{h(x): x \in a\}$, $h(c) = \max\{h(x): x \in c\}$, $h(p) = \max\{h(a_p), h(c_p)\}$. Define $p(\beta)$ so that if $x \in a_{p(\beta)} \cup c_{p(\beta)}$ and $h(x) < \sup\{h(p(\beta')): \beta' < \beta\}$, then $x \in r_a \cup r_c$.

All the above refining has achieved: If $\beta' < \beta$, and $p(\beta)$ and $p(\beta')$ are incompatible, then there is $x \in c_{p(\beta)} - r_c$ such that $b_{p(\beta)}(x) \cap a_{p(\beta')} \neq \varnothing$. List each $c_{p(\beta)}$ as $(x(\beta, j))_{j<k}$. We define a function f_0: $[\omega_1]^2 \to \{0, 1\}$ by $f(\beta', \beta) = 0$ iff $b_{p(\beta)}(x(\beta, 0)) \cap a_{p(\beta')} = \varnothing$. We apply $\omega_1 \to (\omega_1, \omega + 1)^2_2$ to get either S_0 or S_1.

We claim that we get S_0 of order type ω_1. Aiming for a contradiction, assume $S_1 \subset W_4$ has order type $\omega + 1$ and $f(\beta', \beta) = 1$ for all $(\beta', \beta) \in [S_1]^2$. Let $(\gamma_n)_{n\leq\omega}$ enumerate S_1 in increasing order. Let $x(\gamma_\omega, 0) = (m + 1, \lambda)$. Let $(m, \delta_n) \in b_{p(\gamma_\omega)} \cap a_{p(\gamma_n)}$. Then $(\delta_n)_{n\in\omega}$ is an increasing sequence of elements of L_λ whose sup is less than λ. Contradiction.

Define f_1: $[S_0]^2 \to \{0, 1\}$ as we did f_0 with $x(\beta, 0)$ replaced by $x(\beta, 1)$. The same argument gives that we get a new S_0 (rather than S_1). We repeat, applying $\omega_1 \to (\omega_1, \omega + 1)^2_2$ k times, to get $S^* \subset \omega_1$ of order type ω_1; $\{p(\beta): \beta \in S^*\}$ is an

[2]Not exactly. Some unidentified person noted that some proofs of the Dushnik-Miller theorem actually give this slightly stronger result. See [W, Theorem 7.4.1]; [F, Theorem 4.5]. It is also a special case of Corollary 1 to Theorem 3.4, [ER].

uncountable subset of W of compatible elements. We have shown that P_n has property K.

Let P be the product poset $\Pi_n P_n$. We consider P to be the set of functions f with domain ω such that $f(n) \in P_n$ and $\{n \in \omega: f(n) \neq (\emptyset, \emptyset, \emptyset)\}$ is finite; $f' < f$ iff for all $n, f'(n) < f(n)$. The following subsets of P are dense: for all $x \in S(\mathfrak{L})$,

$$D_x = \{f \in P: \exists n\, x \in a_{f(n)}\}$$

for all $n \in \omega$, $x \in H_n$

$$D_{x,n} = \{f \in P: x \in c_{f(n)}\}$$

for all $n \in \omega$, $x \in S(\mathfrak{L})$

$$D'_{x,n} = \{f \in P: x \in c_{f(n)} \cup a_{f(n)}\}.$$

Since there are ω_1 of the above dense subsets, by MA + $\neg$CH there is a filter G on P meeting all of them. For each $n \in \omega$, define $U_n = \bigcup \{c_{f(n)}: f \in G\}$. Then $(U_n)_{n\in\omega}$ is a sequence of open subsets with empty intersection satisfying $U_n \supset H_n$. (The first family of dense sets ensures that $\bigcap_{n\in\omega} U_n = \emptyset$; the second that $U_n \supset H_n$; the third that U_n is open.) We have shown that $S(\mathfrak{L})$ is countably metacompact.

ADDENDA. 1. A simple modification of the proof shows that, assuming MA_{ω_1}, every subset of $S(\mathfrak{L})$ is G_δ.

2. P. Nyikos has also asked whether the following statement, Σ, is a theorem of ZFC.

> "There is a ladder system $\mathfrak{L} = (L_\lambda: \lambda \in \omega_1 \cap \mathrm{LIM})$ such that whenever x is a stationary subset of ω_1, then there is $\lambda < \omega_1$ such that $x \cap L_\lambda$ is infinite."

Negative answers have been given by Komj'ath, Kunen, and the author. Kunen showed that $\mathrm{MA}_{\omega_1} \to \neg\Sigma$, using the product of ω copies of Wage's poset [Wa]. By a density argument, the union of the countably many "generic" sets is ω_1, so at least one must be stationary. The author's negative answer is now a technique looking for an application. The key idea is that the following statement is consistent with ZFC.

> "MA + $2^\omega = \omega_3$ + there is a family $\mathcal{C}$ of club subsets of ω_1 satisfying
> 1. $\mathcal{C}$ has cardinality ω_2.
> 2. If C is club in ω_1, then there is $C' \in \mathcal{C}$, $C' \subset C$."

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