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ON THE LATTICE OF f-PROXIMITIES

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ABSTRACT. Let (X, c) be a Čech closure space. By $\mathfrak{M}_f(X, c)$, we denote the family of all f-proximities on X which induce c. Under suitable restrictions on f, it is proved that $(\mathfrak{M}_f(X, c), \subset)$ is a complete distributive lattice.

1. Introduction. Generalised proximity structures have recently been investigated by Lodato [6], Harris [5], Gagrat and Naimpally [4], Sharma and Naimpally [9], Thron and Warren [11], [12] and others. Thron [10] introduced the notion of f-proximities and has shown that for different choices of f one obtains many of the known types of proximities including among others the S-proximities, LO-proximities, RI-proximities, EF-proximities as well as basic proximities of Čech.

In this article we present a more complete order structure of f-proximities compatible with a given closure space (denoted by $\mathfrak{M}_f(X, c)$). Some of the results obtained in this article overlap those of Dooher and Thron [3], Sharma and Naimpally [9], Thron and Warren [11], [12] and Thron [10].

2. Preliminaries. In this section we fix our notations, collect several definitions and state some results without proofs.

In what follows there is always an underlying nonempty set X. It will be convenient to denote the elements of X by $x, y, \ldots,$ its subsets by $A, B, \ldots$. Families of subsets will be denoted by $\mathfrak{A}, \mathfrak{B}, \ldots$. In particular, $\mathfrak{F}$ will be used for filters, $\mathfrak{U}$, $\mathfrak{B}$ for ultrafilters and $\mathfrak{G}$ for grills. The collection of all grills on X will be denoted by $\Gamma(X)$. However, for proximities we shall use the symbol Π.

We begin by recalling the definition of a grill. Basic results on grills are given in Thron [10].

DEFINITION 2.1. A family $\mathfrak{G}$ of subsets of X is called a *grill* if it satisfies $A \supset B \in \mathfrak{G} \Rightarrow A \in \mathfrak{G}, A \cup B \in \mathfrak{G} \Rightarrow A \in \mathfrak{G}$ or $B \in \mathfrak{G}$ and $\varnothing \notin \mathfrak{G}$.

DEFINITION 2.2. A function $c\colon \mathfrak{P}(X) \to \mathfrak{P}(X)$ is called a *Čech closure operator* on X if it satisfies the following axioms.

C_1: $c(\varnothing) = \varnothing$.

C_2: for every $A \subset X, A \subset c(A)$.

C_3: for all $A, B \subset X$, $c(A \cup B) = c(A) \cup c(B)$.

The pair (X, c) is called a *closure space*. A closure space (X, c) is called R_0 iff $x \in c(y)$ implies $y \in c(x)$ for $x, y \in X$. It is said to be R_1 iff for $x \in X, A \subset X$, $c(x) \cap c(A) = \varnothing$ implies $x \in c(A)$.

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DEFINITION 2.3. *A basic proximity or simply a proximity* Π on a set X is a relation on $\mathfrak{P}(X)$ satisfying the following.

P_1: $\Pi = \Pi^{-1}$.

P_2: $\Pi(A)$ is a grill on X for all $A \subset X$.

P_3: $\Pi(A) \supset \bigcup [\mathfrak{U}: \mathfrak{U}$ is an ultrafilter on X and $A \in \mathfrak{U}]$.

The pair (X, Π) is called a *proximity space*.

For each proximity Π one defines

$$c_\Pi(A) = \big[x: ([x], A) \in \Pi\big] \quad \text{for all } A \subset X.$$

c_Π is called the closure operator induced by Π. Moreover, c_Π is an R_0 closure operator on X.

DEFINITION 2.4. Let $\mathfrak{A}$ be the family of all ordered pairs $(\Pi, \mathfrak{G})$ where Π is a proximity and $\mathfrak{G}$ is a grill on the same set. By a *grill operator* f, we shall mean a function defined on $\mathfrak{A}$ and satisfying

(i) $f(\Pi, \mathfrak{G}) \in \Gamma(X(\Pi))$ and

(ii) $f(\Pi, \mathfrak{G}_1) \subset f(\Pi, \mathfrak{G}_2)$ if $\mathfrak{G}_1 \subset \mathfrak{G}_2$, where $X(\Pi)$ denotes the reference set of Π.

Π is called an *f-proximity* if it satisfies the condition $f(\Pi, \Pi(A)) \subset \Pi(A)$ for all $A \subset X(\Pi)$.

DEFINITION 2.5. A grill operator f is said to be in the *class* A_1 ($f \in A_1$) iff

$$f(\Pi, \mathfrak{G}_1 \cup \mathfrak{G}_2) \subset f(\Pi, \mathfrak{G}_1) \cup f(\Pi, \mathfrak{G}_2)$$

for all proximities Π and for all grills $\mathfrak{G}_1, \mathfrak{G}_2$ on the same set.

The *class* A_2 consists of all grill operators f for which $f(\Pi, \bigcup_{i \in I} \mathfrak{G}_i) \subset \bigcup [f(\Pi, \mathfrak{G}_i): i \in I]$ is valid for all proximities Π and for an arbitrary collection $[\mathfrak{G}_i: i \in I]$ of grills on the same set.

The *class* M consists of grill operators f for which $f(\Pi_1, \mathfrak{G}) \subset f(\Pi_2, \mathfrak{G})$ is valid for all grills $\mathfrak{G}$ and for all proximities Π_1, Π_2 on the same set satisfying $\Pi_1 \subset \Pi_2$.

The grill operator f is said to be in the *class* I iff $f(\Pi_1, \mathfrak{G}) = f(\Pi_2, \mathfrak{G})$ for all grills $\mathfrak{G}$ and for all proximities Π_1, Π_2 on the same set satisfying $c_{\Pi_1} = c_{\Pi_2}$.

REMARK 2.6. The knowledgeable reader will observe that there is a change in our definitions of the grill operator f, f-proximity and the classes A_1, A_2, M and I from Thron's original definitions in **[10]**.

A large number of grill operators beyond the six (i, r, b, h, e, s) originally given by Thron **[10]** are known. These can be found in R. Ori **[7]** among others. For r and h we list the formulations below rather than the original definitions in **[10]**.

$$r(\Pi, \mathfrak{G}) = \big[B \subset X(\Pi): \exists x \in X(\Pi), [x] \in \Pi(A) \cap \mathfrak{G}\big];$$

$$h(\Pi, \mathfrak{G}) = \big[B \subset X(\Pi): \exists b \in B \text{ and } \mathfrak{U} \subset \Pi([b]) \cap \mathfrak{G}\big].$$

We also list the definition of t as given in **[7]**.

$$t(\Pi, \mathfrak{G}) = \big[B \subset X(\Pi): O_0(\Pi, B) \subset \mathfrak{G}\big],$$

where $O_0(\Pi, B) = [E \subset X(\Pi): E \supset B, C_\Pi(X - E) = X - E]$. It may be verified that

(i) $[i, b, r] \subset A_2 \subset A_1$,

(ii) $[t, h, e] \subset A_1$,

(iii) $[i, r, b, h] \subset I$,
(iv) $[i, r, b, h, e, t] \subset M$.
Set $\mathfrak{M}_f(X, c) = [\Pi\colon \Pi$ is an f-proximity on X and $c_\Pi = c]$.

3. The order structure of $(\mathfrak{M}_f(X, c), \subset)$. For the case $f = e$ studies in this direction go back to Dooher and Thron [3], whereas for $f = i$ and $f = b$, Thron and Warren [10], [11] have investigated the order structure of $(\mathfrak{M}_f(X, c), \subset)$. Their success has led us to the investigations described below. The main results of Thron and Warren [10], [11] are special cases of our main theorem of this section. The knowledgeable reader will observe that the main theorem also generalises a result of Thron [10, Theorem 7.7.9].

The collection $\mathfrak{M}(X)$ of all proximities on X with respect to the set inclusion is a partially ordered set. The elements $[(A, B)\colon A, B \subset X, A \cup B \neq \varnothing]$ and $[(A, B)\colon A \neq \varnothing, B \neq \varnothing]$, respectively, are the least and the largest elements of $(\mathfrak{M}(X), \subset)$. Moreover, $\bigcup[\Pi\alpha\colon \alpha \in \Gamma]$, where $[\Pi\alpha\colon \alpha \in \Gamma]$ is the nonempty subfamily of $\mathfrak{M}(X)$, is the least upper bound of the nonempty subfamily in $\mathfrak{M}(X)$. Hence $(\mathfrak{M}(X), \subset)$ is a complete lattice.

Throughout, $\bigwedge[\Pi\alpha\colon \alpha \in \Gamma]$ and $\bigvee[\Pi\alpha\colon \alpha \in \Gamma]$ will denote, respectively, the glb and the lub of $[\Pi\alpha\colon \alpha \in \Gamma]$ in the lattice $(\mathfrak{M}(X), \subset)$. Theorem 3.1 below contains a description of the glb, $\bigwedge[\Pi\alpha\colon \alpha \in \Gamma]$, of a nonempty subfamily $[\Pi\alpha\colon \alpha \in \Gamma]$ of $\mathfrak{M}(X)$.

THEOREM 3.1. *Let* $[\Pi\alpha\colon \alpha \in \Gamma]$ *be a nonempty subfamily of* $\mathfrak{M}(X)$. *Define a relation* Π *on* $\mathfrak{P}(X)$ *as follows.* $(A, B) \in \Pi$ *iff given finite cover* $[A_1, A_2, \ldots, A_n]$ *of* A *and* $[B_1, B_2, \ldots, B_m]$ *of* B, *there exists* A_i $(1 \leqslant i \leqslant n)$ *and* B_j $(i \leqslant j \leqslant m)$ *such that* $(A_i, B_j) \in \Pi\alpha$ *for all* $\alpha \in \Gamma$. *Then* Π *is a proximity on* X *and*

$$\Pi = \bigwedge[\Pi\alpha\colon \alpha \in \Gamma].$$

Moreover, if $c_{\Pi\alpha} = c$ *for all* $\alpha \in \Gamma$, *then* $c_\Pi = c$.

The details of the proof of Theorem 3.1 are omitted since they can be found in Čech [2].

THEOREM 3.2. *The lattice* $(\mathfrak{M}(X), \subset)$ *is distributive.*

The details of the proof of Theorem 3.2 can be found in Thron and Warren [11].

PROPOSITION 3.3. *Let* f *be a grill operator in the class* $A_1 \cap (M \cup I)$ *and* $[\Pi\alpha\colon \alpha \in \Gamma]$ *be a nonempty subfamily of* $\mathfrak{M}_f(X, c)$. *Then* $\bigwedge[\Pi\alpha\colon \alpha \in \Gamma] \in \mathfrak{M}_f(X, c)$. *In particular* $(\mathfrak{M}_f(X, c), \subset)$ *has the least element if* $\mathfrak{M}_f(X, c)$ *is nonempty.*

PROOF. Let $\Pi = \bigwedge[\Pi\alpha\colon \alpha \in \Gamma]$, where $\Pi\alpha \in \mathfrak{M}_f(X, c)$ for each $\alpha \in \Gamma$. In view of Theorem 3.1, the proof will be completed if we check that $f(\Pi, \Pi(D)) \subset \Pi(D)$ for each $D \in \mathfrak{P}(X)$. Let $B \in f(\Pi, \Pi(D))$. Suppose $[B_1, B_2, \ldots, B_m]$ and $[D_1, D_2, \ldots, D_n]$, respectively, are finite covers of B and D. Since $f(\Pi, \Pi(D))$ is a grill and $f \in A_1$ it follows that there exists B_j $(1 \leqslant j \leqslant m)$ and D_i $(i \leqslant i \leqslant n)$ such that $B_j \in f(\Pi, \Pi(D_i))$.

Since $\Pi(D_i) \subset \Pi\alpha(D_i)$ for each $\alpha \in \Gamma$, by the monotonicity of the grill operator, it follows that $B_j \in f(\Pi, \Pi\alpha(D_i))$ for each $\alpha \in \Gamma$. Thus we have provided the following.

For $f \in A_1$, $B \in f(\Pi, \Pi(D))$ and finite covers $[B_1, B_2, \ldots, B_m]$ of B and $[D_1, D_2, \ldots, D_n]$ of D, there exist B_j $(1 < j < m)$ and D_i $(1 < i < n)$ such that

$$B_j \in f(\Pi, \Pi\alpha(D_i)) \quad \text{for each } \alpha \in \Gamma. \tag{1}$$

Case I. $f \in A_1 \cap M$. Since $f \in M$ and $\Pi \subset \Pi\alpha$ for all $\alpha \in \Gamma$ it follows that

$$f(\Pi, \Pi\alpha(D_1)) \subset f(\Pi\alpha, \Pi\alpha(D_i)) \qquad \forall \alpha \in \Gamma.$$

Since $\Pi\alpha$ is an f-proximity we have, using (1), that $B_j \in \Pi\alpha(D_i)$ for each $\alpha \in \Gamma$. Thus $B \in \Pi(D)$ and hence $f(\Pi, \Pi(D)) \subset \Pi(D)$.

Case II. $f \in A_1 \cap I$. Since $f \in I$ and each $\Pi\alpha \in \mathfrak{M}_f(X, c)$ and $c_{\Pi\alpha} = c_\Pi = c$, it follows that

$$f(\Pi, \Pi\alpha(D_i)) = f(\Pi\alpha, \Pi\alpha(D_i)) \subset \Pi\alpha(D_i)$$

for each $\alpha \in \Gamma$. The argument is now completed in the same way as in Case I.

This completes the proof.

Proposition 3.4. *Suppose $f \in A_2 \cap I$. Let $[\Pi\alpha: \alpha \in \Gamma]$ be a nonempty subfamily of $\mathfrak{M}_f(X, c)$. Then $\bigvee[\Pi\alpha: \alpha \in \Gamma] \in \mathfrak{M}_f(X, c)$.*

Proof. We know that $\bigvee[\Pi\alpha: \alpha \in \Gamma] = \bigcup\, [\Pi\alpha: \alpha \in \Gamma]$. Write

$$\Pi = \bigcup\, [\Pi\alpha: \alpha \in \Gamma].$$

Clearly $c_\Pi = c$. To complete the proof we need to check that $f(\Pi, \Pi(A)) \subset \Pi(A)$ for each $A \subset X$. Let $B \in f(\Pi, \Pi(A)) = f(\Pi, \bigcup_{\alpha\in\Gamma} \Pi\alpha(A))$. Since $f \in A_2$, it follows that $\exists \alpha \in \Gamma$ such that $B \in f(\Pi, \Pi\alpha(A))$. Using the fact that $f \in I$, we have $B \in f(\Pi\alpha, \Pi\alpha(A)) \subset \Pi\alpha(A)$ as $\Pi\alpha \in \mathfrak{M}_f(X, c)$. Thus $B \in \Pi(A)$.

This completes the proof.

Theorem 3.5. *Let $f \in A_1 \cap I$. Then $(\mathfrak{M}_f(X, c), \subset)$ is a distributive lattice. In particular, $(\mathfrak{M}_h(X, c), \subset)$ is a distributive lattice.*

Proof. For any nonempty subfamily $[\Pi\alpha: \alpha \in \Gamma]$ of $\mathfrak{M}_f(X, c)$, the glb, $\bigwedge[\Pi\alpha: \alpha \in \Gamma]$, belongs to $\mathfrak{M}_f(X, c)$. This, in fact, is the conclusion of Proposition 3.3. In particular, $\Pi_1 \wedge \Pi_2 \in \mathfrak{M}_f(X, c)$ for any two $\Pi_1, \Pi_2 \in \mathfrak{M}_f(X, c)$.

Let Π_1, Π_2 be any two elements in $\mathfrak{M}_f(X, c)$. We know that $\Pi_1 \vee \Pi_2 = \Pi_1 \cup \Pi_2$ is a proximity on X such that $c_{\Pi_1 \cup \Pi_2} = c$. Also for any $A \subset X$, we have

$$\begin{aligned} f(\Pi_1 \vee \Pi_2, (\Pi_1 \vee \Pi_2)(A)) &= f(\Pi_1 \cup \Pi_2, \Pi_1(A) \cup \Pi_2(A)) \\ &\subset f(\Pi_1, \Pi_1(A)) \cup f(\Pi_2, \Pi_2(A)) \\ &\subset \Pi_1(A) \cup \Pi_2(A) = (\Pi_1 \vee \Pi_2)(A), \end{aligned}$$

Since $f \in A_1 \cap I$ and Π_1, Π_2 are f-proximities. Thus $\Pi_1 \vee \Pi_2 \in \mathfrak{M}_f(X, c)$. It may be pointed out that the glb and lub of any two elements in $\mathfrak{M}_f(X, c)$ are the same as in the lattice $(\mathfrak{M}(X), \subset)$. Consequently, it follows that $(\mathfrak{M}_f(X, c), \subset)$ is a sublattice of the distributive lattice $(\mathfrak{M}(X), \subset)$ and hence is itself distributive.

Combining Propositions 3.3, 3.4, 3.5 and using the fact that $A_2 \subset A_1$, we obtain our main result.

THEOREM 3.6. *Let* $f \in A_2 \cap I$. *Suppose that* $\mathfrak{M}_f(X, c) \neq \varnothing$. *Then* $(\mathfrak{M}_f(X, c), \subset)$ *is a complete distributive lattice. Moreover,*

$$\bigwedge \left[\Pi\alpha \colon \Pi\alpha \in \mathfrak{M}_f(X, c)\right] \quad \textit{and} \quad \bigvee \left[\Pi\alpha \colon \Pi\alpha \in \mathfrak{M}_f(X, c)\right]$$

are the least and the largest elements of $(\mathfrak{M}_f(X, c), \subset)$, *respectively.*

REMARK. Since $[i, r, b] \subset A_2 \cap I$, it follows that the above theorem holds for $f = i, r$ and b. Ori [7] has proven that $\mathfrak{M}_e(X, c)$ has a largest element iff (X, c) is completely regular and locally compact; each $\mathfrak{M}_t(X, c)$ and $\mathfrak{M}_h(X, c)$ has a largest element iff (X, c) is regular and locally compact. We thus have the following.

PROPOSITION 3.7. (a) *If* (X, c) *is regular and locally compact, then* $(\mathfrak{M}_h(X, c), \subset)$ *is a complete distributive lattice.*

(b) *If* (X, c) *is regular and locally compact, then* $(\mathfrak{M}_t(X, c), \subset)$ *is a complete lattice.*

(c) *If* (X, c) *is completely regular and locally compact, then* $(\mathfrak{M}_e(X, c), \subset)$ *is a complete lattice.*

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