EXPONENTIAL SUMS RELATED TO BINOMIAL COEFFICIENT PARITY

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ABSTRACT. Let $\alpha(n)$ be the number of 1's in the binary expansion of n , $z > 0$ and $\phi_z(x) = \sum_{n < x} z^{\alpha(n)}$. Let $\theta_z = (\log(1+z))/\log 2$, $a(z) = \liminf x^{-\theta_z} \phi_z(x)$, $b(z) = \limsup x^{-\theta_z} \phi_z(x)$. Then $0 < a(z) < 1 < b(z) < 2$. Furthermore, if $z \neq 1$, then $a(z) < b(z)$.

1. Let $F_1(x)$ be the number of odd integers in the first x rows of Pascal's triangle. The behavior of F_1 has been studied in [4] and [7] and has been found to be somewhat erratic.

If we let $\beta(n)$ be the number of odd integers in the n th row of Pascal's triangle, then it can be shown that $\beta(n) = 2^{\alpha(n-1)}$, where $\alpha(n)$ is the number of 1's in the binary expansion of n . (In fact, $\binom{n}{j}$ is odd precisely when the places where the binary expansion of j has 1's are places where the binary expansion of n has 1's. This result was shown by Glaisher [3].) We thus observe that

$$F_1(x) = \sum_{n < x} \beta(n) = \sum_{n < x} 2^{\alpha(n)}. \quad (1.1)$$

Let us more generally define

$$F_c(x) = \sum_{n < x} \beta^c(n). \quad (1.2)$$

Then $F_c(x) = \sum_{n < x} (2^{\alpha(n)})^c = \sum_{n < x} (2^c)^{\alpha(n)}$.

This suggests that we study sums of the form

$$\phi_z(x) = \sum_{n < x} z^{\alpha(n)}, \quad z > 0. \quad (1.3)$$

As one might suspect, these functions exhibit a behavior similar to that of F_1 .

Note. We will generally write $\phi(x)$ rather than $\phi_z(x)$.

2. We begin by evaluating $\phi(x)$ when x is a power of 2. Since there are $\binom{n}{k}$ integers m less than 2^n with k ones in their binary expansions,

$$\phi(2^n) = \sum_{m < 2^n} z^{\alpha(m)} = \sum_{k < n} z^k \binom{n}{k} = (1+z)^n. \quad (2.1)$$

We thus have proven

LEMMA 2.1. $\phi(2^n) = (1+z)^n$.

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If we let $\theta = \theta_z = (\log(1+z))/\log 2$, then we can write $\phi(2^n) = (2^n)^\theta$. Stolarsky [7] showed that for $z = 2$, $\phi(x)$ is of the order x^θ . We shall show that result holds for all z by using the following lemma.

LEMMA 2.2. *If $0 < x < 2^n$, then*

$$\phi(2^n + x) = \phi(2^n) + z\phi(x). \quad (2.2)$$

PROOF. First note that $\phi(2^n + x) = \sum_{m < 2^n} z^{\alpha(m)} + \sum_{m < x} z^{\alpha(2^n + m)}$. Since the binary expansions of 2^n and m do not overlap if $m < 2^n$, $\alpha(2^n + m) = 1 + \alpha(m)$ and hence $z^{\alpha(2^n + m)} = z \cdot z^{\alpha(m)}$. Thus

$$\phi(2^n + x) = \sum_{m < 2^n} z^{\alpha(m)} + z \sum_{m < x} z^{\alpha(m)} = \phi(2^n) + z\phi(x). \quad (2.3)$$

We can now prove

THEOREM 2.3. $x^\theta/(1+z) < \phi(x) \leq (1+z)x^\theta$.

PROOF. Let $x = 2^n + y$, $y < 2^n$. Then

$$\phi(x) = \phi(2^n) + z\phi(y) \leq (1+z)\phi(2^n) = (1+z)(2^n)^\theta < (1+z)x^\theta. \quad (2.4)$$

Similarly,

$$\phi(x) > \phi(2^n) = (2^n)^\theta > (2^{n+1})^\theta/2^\theta > x^\theta/(1+z). \quad (2.5)$$

If $z < 1$, then the lower bound can be improved to x^θ , while if $z > 1$, then the upper bound can be improved to x^θ .

THEOREM 2.4. *If $z > 1$, then $\phi(x) \leq x^\theta$; if $z < 1$, then $\phi(x) \geq x^\theta$.*

PROOF. Assume $z > 1$ and $\phi(x) \leq x^\theta$ for all $x \leq 2^n$. If $2^n < x \leq 2^{n+1}$, then $x = 2^n + y$ for $y \leq 2^n$.

Hence

$$\frac{\phi(x)}{x^\theta} = \frac{\phi(2^n) + z\phi(y)}{(2^n + y)^\theta} \leq \frac{(1+z)^n + zy^\theta}{(2^n + y)^\theta}. \quad (2.6)$$

Let $g(y) = ((1+z)^n + zy^\theta)/(2^n + y)^\theta$ for $y \geq 0$. Then

$$g'(y) = \theta \{2^n zy^{\theta-1} - (1+z)^n\} / (2^n + y)^{\theta+1}. \quad (2.7)$$

Since $\theta > 1$, there exists some positive y_0 such that $g'(y) < 0$ if $y < y_0$ and $g'(y) > 0$ if $y > y_0$. Thus any maximum for g on $[0, 2^n]$ must occur at either 0 or 2^n . Since $g(0) = g(2^n) = 1$, $g(y) \leq 1$ on $[0, 2^n]$ and hence $\phi(x) \leq x^\theta$ for $x \leq 2^{n+1}$. The proof follows by induction.

A similar proof holds if $z < 1$. In this case assume $\phi(x) \geq x^\theta$ if $x \leq 2^n$. Write $x = 2^n + y$ as before for $2^n < x \leq 2^{n+1}$. In place of (2.6) we have

$$\frac{\phi(x)}{x^\theta} \geq \frac{(1+z)^n + zy^\theta}{(2^n + y)^\theta} = g(y). \quad (2.8)$$

Since $\theta < 1$ when $z < 1$, there exists some positive y_0 such that $g'(y) > 0$ if $0 < y < y_0$ and $g'(y) < 0$ if $y > y_0$. Thus in this case any minimum for g on $[0, 2^n]$ must occur at either 0 or 2^n . Again $g(0) = g(2^n) = 1$ implies $g(y) \geq 1$ for $0 < y < 2^n$ and hence $\phi(x) \geq x^\theta$ for $x \leq 2^{n+1}$. Q.E.D.

Note. The proof actually shows that the equalities in the statement of Theorem 2.4 hold only when x is a power of 2.

3. Thus far we have demonstrated a degree of regularity in the behavior of $\phi(x)$. We now demonstrate some erratic behavior of $\phi(x)$. Let us define $\psi(x) = \psi_z(x) = \phi(x)/x^\theta$. We have shown that if $z < 1$, then $1 < \psi(x) < 1 + z$, while if $z > 1$, then $1/(1 + z) < \psi(x) < 1$. We now show

THEOREM 3.1. *If $z \neq 1$, then $\lim_{x \rightarrow \infty} \psi(x)$ does not exist.*

We prove this theorem via two lemmas.

LEMMA 3.2. *Let $z > 1$. Then either $\psi(2x + 1) < \psi(x)$ or $\psi(2x - 1) < \psi(x)$.*

LEMMA 3.3. *Let $z < 1$, x odd. Then either $\psi(2x + 1) > \psi(x)$ or $\psi(2x - 1) > \psi(x)$.*

We first make the observation that $\psi(2x) = \psi(x)$, i.e. $\phi(2x) = (1 + z)\phi(x)$. Observe that

$$\phi(2x) = \sum_{m < 2x} z^{\alpha(m)} = \sum_{m < x} \{z^{\alpha(2m)} + z^{\alpha(2m+1)}\}. \quad (3.1)$$

Since $\alpha(2m) = \alpha(m)$ and $\alpha(2m + 1) = 1 + \alpha(m)$, (3.1) yields

$$\phi(2x) = \sum_{m < x} z^{\alpha(m)}(1 + z) = (1 + z)\phi(x). \quad (3.2)$$

Now consider $\phi(2x + 1)$ and $\phi(2x - 1)$.

$$\phi(2x + 1) = \phi(2x) + \beta(2x + 1) = \phi(2x) + z^{\alpha(2x)}. \quad (3.3)$$

Since $\alpha(2x) = \alpha(x)$ and $\phi(2x) = (1 + z)\phi(x)$,

$$\phi(2x + 1) = (1 + z)\phi(x) + z^{\alpha(x)}. \quad (3.4)$$

Similarly,

$$\phi(2x - 1) = \phi(2x) - \beta(2x) = \phi(2x) - z^{\alpha(2x-1)}. \quad (3.5)$$

Since $\alpha(2x - 1) > \alpha(x)$ with equality holding if x is odd, (3.5) yields

$$\phi(2x - 1) < (1 + z)\phi(x) - z^{\alpha(x)} \quad (3.6)$$

with equality holding if x is odd.

Adding (3.4) and (3.6) together yields

$$\phi(2x + 1) + \phi(2x - 1) < 2(1 + z)\phi(x) \quad (3.7)$$

with equality holding if x is odd. We are now in a position to prove the lemmas.

First let $z > 1$ and assume $\psi(2x + 1) > \psi(x)$ and $\psi(2x - 1) > \psi(x)$. Then

$$\phi(2x + 1) > \phi(x)(2x + 1)^\theta / x^\theta = \phi(x)(1 + z)(1 + 1/2x)^\theta \quad (3.8)$$

and similarly

$$\phi(2x - 1) > \phi(x)(2x - 1)^\theta / x^\theta = \phi(x)(1 + z)(1 - 1/2x)^\theta. \quad (3.9)$$

Thus

$$\phi(2x + 1) + \phi(2x - 1) > \phi(x)(1 + z)\{(1 + 1/2x)^\theta + (1 - 1/2x)^\theta\}. \quad (3.10)$$

Comparing (3.7) and (3.10) yields

$$(1 + 1/2x)^\theta + (1 - 1/2x)^\theta < 2 \quad (3.11)$$

which is false since $\theta > 1$. That proves Lemma 3.2.

To prove Lemma 3.3, recall that if x is odd, then equality holds in (3.7). Also, if $\psi(2x + 1) < \psi(x)$ and $\psi(2x - 1) < 1$, then the inequalities in (3.8)–(3.10) reverse and thus (3.11) becomes

$$(1 + 1/2x)^\theta + (1 - 1/2x)^\theta > 2. \quad (3.12)$$

However, if $z < 1$, then $\theta < 1$ and (3.12) is false, proving Lemma 3.3. We note that in Lemma 3.2, $2x \pm 1$ may be replaced by $2^r x \pm 1$ for any positive integer r .

Theorem 3.1 follows immediately. If $z > 1$, Lemma 3.2 shows that for any fixed n we can find a sequence $\{n_r\}$ with $n_{r+1} = 2n_r \pm 1$ such that $\psi(n_{r+1}) < \psi(n_r)$. Then $\liminf \psi(x) \leq \lim_{r \rightarrow \infty} \psi(n_r) < \psi(n_0) < 1$, whereas $\limsup \psi(x) = 1$ since $\psi(x) < 1$ and $\psi(2^n) = 1$ for all n . The case $z < 1$ is handled similarly.

The case $z = 2$ was considered by Harborth [4], who conjectured that if $n_0 = 1$ and $n_{r+1} = 2n_r \pm 1$ is chosen so as to minimize $\psi(n_{r+1})$ locally, then $\lim \psi(n_r) = \liminf \psi(x)$ and showed that they agreed to six decimal places. In the next section we show how $\liminf \psi(x)$ can be approximated as accurately as desired. We also suspect that Harborth's conjecture is valid for all $z \neq 1$.

4. We demonstrate the technique for $z = 2$, but the method will clearly hold for all $z \neq 1$. First observe that (3.4) yields $\phi(2x + 1) > 3\phi(x)$ and hence

$$\psi(2x + 1) > 3\phi(x)/(2x + 1)^\theta = \psi(x)/(1 + 1/2x)^\theta. \quad (4.1)$$

If $2^n < x < 2^{n+1}$, then (4.1) implies that

$$\psi(2x + 1) > \psi(x)/(1 + 1/2^{n+1})^\theta. \quad (4.2)$$

Suppose $x > 2^n$. Then there is a sequence $x_0, x_1, \dots, x_r$ such that $2^{n+i} < x_i < 2^{n+i+1}$, $x_{i+1} = 2x_i + \varepsilon$ for $\varepsilon = 0$ or 1 and $x_r = x$. Since $\psi(2x) = \psi(x)$, repeated application of (4.2) yields

$$\psi(x) > \psi(x_0)/P^\theta \quad (4.3)$$

where $P = \prod_{i=1}^{r+1} (1 + 1/2^{n+i})$. We need an upper bound for P . Observe that $\log P = \sum_{i=1}^{r+1} \log(1 + 1/2^{n+i}) < \sum_{i \geq n} (1/2^i) = 1/2^n$ so $P < e^{1/2^n}$ and $P^\theta < e^{\theta/2^n}$. We find

$$\psi(x) > \psi(x_0)/e^{\theta/2^n}. \quad (4.4)$$

If we let $q_n = \min_{2^n < x < 2^{n+1}} \psi(x)$, then (4.4) yields

$$\psi(x) > q_n / \exp(\theta/2^n) = q_n - q_n(1 - 1/\exp(\theta/2^n)) \quad (4.5)$$

and hence (since $q_n < 1$)

$$\psi(x) > q_n - (1 - 1/\exp(\theta/2^n)). \quad (4.6)$$

We can obviously make $1 - 1/\exp(\theta/2^n)$ arbitrarily small merely by making n large enough. In particular, $1 - 1/\exp(\theta/2^n) < \varepsilon$ if $n > \log(\theta/\varepsilon)/\log 2$, i.e. if $2^n > \theta/\varepsilon$. In particular, this technique shows that q_{22} should agree with

$\liminf \psi(x)$ to six decimal places. In [4], Harborth showed that q_{19} agrees with $\liminf \psi(x)$ to six decimal places.

5. Let $a(z) = \liminf_{x \rightarrow \infty} \psi_z(x)$, $b(z) = \limsup_{x \rightarrow \infty} \psi_z(x)$. We have already shown that if $z < 1$, then $a(z) = 1 < b(z)$ while if $z > 1$, then $a(z) < 1 = b(z)$. We now exhibit the asymptotic behavior of $a(z)$ and $b(z)$.

THEOREM 5.1. $\lim_{z \rightarrow \infty} a(z) = 0$, $\lim_{z \rightarrow 0} b(z) = 1$.

The proof is direct. The first part follows from

$$0 < a(z) < \psi_z(3) = \frac{1 + 2z}{3^{\theta_z}} = \frac{1 + 2z}{(z + 1)^{\log 3 / \log 2}} \quad (5.1)$$

and $(1 + 2z)/(z + 1)^{\log 3 / \log 2} \rightarrow 0$ as $z \rightarrow \infty$.

The second part is an immediate consequence of Theorem 2.3.

6. We generalize the explicit formula for ϕ_2 given by Stolarsky [7].

THEOREM 6.1. Let $x = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r}$, where $a_1 < a_2 < \cdots < a_r$. Then $\phi(x) = (1 + z)^{a_r} + z(1 + z)^{a_{r-1}} + \cdots + z^{r-1}(1 + z)^{a_1}$.

PROOF. Assume the formula is true for $\alpha(x) < r$. Then by Lemma 2.2,

$$\phi(x) = \phi(2^{a_r}) + z\phi(2^{a_1} + \cdots + 2^{a_{r-1}}). \quad (6.1)$$

The induction implies that

$$\begin{aligned} & z\phi(2^{a_1} + \cdots + 2^{a_{r-1}}) \\ &= z\{(1 + z)^{a_{r-1}} + z(1 + z)^{a_{r-2}} + \cdots + z^{r-2}(1 + z)^{a_1}\}. \end{aligned} \quad (6.2)$$

Combining (6.1) and (6.2) with Lemma 2.1 completes the induction.

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