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ZERO-DIMINISHING LINEAR TRANSFORMATIONS

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ABSTRACT. This paper solves an open problem of Karlin. We consider linear transformations defined on the set of all polynomials with real coefficients and give a complete characterization of the transformations with the property that they maintain or reduce the number of real zeros of each polynomial.

1. Introduction. Let f be any real-valued function defined on $(-\infty, \infty)$. We shall use the notation $Z_R(f)$, $Z_+(f)$ and $Z_-(f)$ for the number of real zeros of f, the number of zeros in $(0, \infty)$ and the number of zeros in $(-\infty, 0)$, respectively, counting multiplicities.

Let T be a linear transformation defined on polynomials by $T(\sum_{k=0}^n a_k x^k) = \sum_{k=0}^n a_k\lambda_k x^k$ with the eigenvalues λ_k real, $k = 0, 1, 2, \ldots$. In [3], Karlin raises the following question: When is a sequence $\lambda_0, \lambda_1, \lambda_2, \ldots$ a zero-diminishing sequence; that is, for which transformations T does $Z_R(Tf) \leq Z_R(f)$ hold for all polynomials f? As a special case, Pólya [8] and Karlin [3, p. 379] characterize the entire functions $\psi(z)$ for which $\lambda_k = 1/\psi(k)$ has this property. This result is based on the following theorem of Laguerre [7, p. 6].

THEOREM 1.1 (LAGUERRE). *Let $Q(x)$ be a polynomial, all of whose zeros are real and negative. Let $F(x) = \sum a_k x^k$ be an arbitrary polynomial. Then*

$$Z_R(\sum a_k Q(k) x^k) \geq Z_R(\sum a_k x^k).$$

For the various analogues and extensions of Theorem 1.1 we refer the reader to [4], [6], [8] and [11]. In addition, an account of these important investigations is given by M. Marden [5].

One notes immediately from the proof of Theorem 1.1 (see, for example, [7, p. 6]) that Z_R can be replaced by Z_+ or Z_- or, in fact, by the number of zeros in an interval $(0, a)$ for any positive number a.

EXAMPLE 1.2. When $Q(x)$ has degree 1, this corresponds to applying the differential operator $(a + xD_x)$, $a > 0$. The corresponding zero-diminishing transformation T is defined by $T(x^k) = x^k/(a + k)$; that is, the eigenvalues are $\lambda_k = (a + k)^{-1}$, $k = 0, 1, 2, \ldots$.

Besides the class of zero-diminishing transformations arising in the above manner, a handful of special cases have been worked out by Pólya [8]. (For additional references and examples see Karlin [3] and Pólya and Szegö [10].) In the

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following section we provide a complete characterization of such linear transformations.

2. The characterization. We begin by noting that if T is a zero-diminishing transformation, then all of the eigenvalues λ_k are nonzero. For if $\lambda_k = 0$, then $T(x^k) = 0$ has infinitely many zeros while x^k has only k including multiplicity.

THEOREM 2.1. *Let the transformation T be defined by $T(\sum_{k=0}^n a_k x^k) = \sum_{k=0}^n a_k \lambda_k x^k$; $\lambda_k \neq 0$, $k = 0, 1, 2, \ldots$. The following statements are equivalent.*

(i) *Either* (a) $Z_+(Tf) \leq Z_+(f)$ *for all polynomials f or* (b) $Z_+(Tf) \leq Z_-(f)$ *for all polynomials f.*

(ii) *T is a zero-diminishing transformation; that is, $Z_R(Tf) \leq Z_R(f)$ for all polynomials f.*

(iii) *For each integer n, the polynomial $\sum_{k=0}^n \binom{n}{k} \lambda_k^{-1} x^k$ has only real zeros, all of the same sign.*

(iv) *The series $\varphi(z) = \sum_{k=0}^\infty (k!\lambda_k)^{-1} z^k$ converges in the whole complex plane and the entire function $\varphi(z)$ can be represented in the form $\varphi(z) = ce^{\sigma z}\prod_{n=1}^\infty (1 + z/z_n)$ where $\sigma \geq 0$, $z_n > 0$, c is real and $\sum_{n=1}^\infty z_n^{-1} < \infty$.*

Furthermore, in case (i)(a) *all of the numbers λ_k have the same sign and in case* (i)(b) *they alternate in sign.*

PROOF. (i) $\Rightarrow$ (ii). If (i)(a) holds, then let $g(x) = f(-x)$. Since $Z_+(Tg) \leq Z_+(g)$, we have $Z_-(Tf) \leq Z_-(f)$. The polynomials Tf and f have the same multiplicity of the origin as a root and $Z_+(Tf) \leq Z_+(f)$ by hypothesis. Thus $Z_R(Tf) \leq Z_R(f)$. If (i)(b) holds, we again set $g(x) = f(-x)$, obtaining $Z_-(Tf) = Z_+(Tg) \leq Z_-(g) = Z_+(f)$, which as before leads to $Z_R(Tf) \leq Z_R(f)$.

(ii) $\Rightarrow$ (iii). In particular, if Tf is a polynomial with only real zeros, the polynomial f must also have only real zeros. Equivalently, for any polynomial $f(x) = \sum a_k x^k$ with only real zeros, the polynomial $T^{-1}f(x) = \sum a_k \lambda_k^{-1} x^k$ also has only real zeros. Thus for any positive integer n, $T^{-1}(x+1)^n = \sum_{k=0}^n \binom{n}{k}\lambda_k^{-1}x^k$ has only real zeros. Considering $T^{-1}x^k(x+1)(x-1) = (\lambda_k^{-1} - \lambda_{k+2}^{-1}x^2)x^k$, we see that the numbers λ_k and λ_{k+2} must have the same sign. If all λ_k have the same sign, the roots of $\sum \binom{n}{k}\lambda_k^{-1}x^k$ are all negative, and if the λ_k alternate in sign, the roots are all positive.

(iii) $\Leftrightarrow$ (iv). This equivalence is a classical result of Pólya and Schur [9].

(iii) $\Rightarrow$ (i). Assume $g_m(x) = \sum_{k=0}^m \binom{m}{k}\lambda_k^{-1}x^k$ has only real zeros, all of the same sign. Then all of the λ_k have the same sign or they alternate in sign. Without loss of generality, we may assume that the zeros of $g_m(x)$ are all negative and thus the λ_k all have the same sign. Otherwise, let $\mu_k = (-1)^k\lambda_k$ and S be the transformation defined by the μ_k, which all have the same sign. If we know $Z_+(Sf(x)) \leq Z_+(f(x))$, then we obtain

$$Z_+(Tf(-x)) = Z_+(Sf(x)) \leq Z_+(f(x)) = Z_-(f(-x)),$$

thus completing the second case since the polynomial f is arbitrary. Clearly we may also assume $\lambda_k > 0$ for each k.

Now let $f(x) = \sum_{k=0}^n a_k x^k$ be an arbitrary polynomial of degree n, and consider the polynomial $g_m(x) = \sum_{k=0}^m \binom{m}{k}\lambda_k^{-1}x^k$, $m \geq n$, which has only real negative zeros.

In [**2**, Corollary 3.6], we have generalized a classical theorem of Schur [**6**, Satz 7.4] to arbitrary polynomials f. This result states that $Z_+(\sum_{k=0}^n k!a_k\binom{m}{k}\lambda_k^{-1}x^k) \geq Z_+(f)$. Replacing x by x/m gives

$$Z_+\left(\sum_{k=0}^n a_k\lambda_k^{-1}\frac{m!}{(m-k)!m^k}x^k\right) \geq Z_+(f), \tag{2.2}$$

for all $m \geq n$. We note that $\lim_{m\to\infty} m!/(m-k)!m^k = 1$ for each fixed k. By Hurwitz's theorem, taking the limit as m approaches infinity cannot decrease the left-hand side of (2.2). Therefore we obtain

$$Z_+\left(\sum_{k=0}^n a_k\lambda_k^{-1}x^k\right) \geq Z_+(f),$$

or equivalently, $Z_+(f) \geq Z_+(Tf)$, thus concluding the proof of the theorem.

Replacing the variable x by e^x in part (i)(a) of the theorem gives the following result on exponential polynomials.

COROLLARY 2.3. *Let $\lambda_0, \lambda_1, \lambda_2, \ldots$ be a zero-diminishing sequence with all $\lambda_k > 0$. For any real numbers $a_0, a_1, \ldots, a_n$, we have*

$$Z_R\left(\sum_{k=0}^n a_k\lambda_k e^{kx}\right) \leq Z_R\left(\sum_{k=0}^n a_k e^{kx}\right).$$

We note that this corollary is closely related to results of Benz [**1**, Satz 7, p. 273].

REMARK 2.4. There is still an open problem regarding finite sequences $\lambda_0, \lambda_1, \ldots, \lambda_n$ which are zero-diminishing for all polynomials of degree at most n. Our result leads only to the sequences given by part (iv) of Theorem 2.1: either they are part of a longer sequence or the polynomial $\sum_{k=0}^n (k!\lambda_k)^{-1}x^k$ must have only real zeros, all of the same sign. One can construct examples to show that other finite zero-diminishing sequences also exist which do not correspond to the initial portion of an entire function in the class considered in (iv) of Theorem 2.1.

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