COHOMOLOGICAL DIMENSION OF AN ABELIAN MONOID

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ABSTRACT. It is shown that the cohomological dimension of an abelian monoid is equal to that of its group reflection provided that the monoid is either finitely generated or cancellative.

Let M be a monoid and let R be a nonzero ring with identity. We shall denote the monoid ring of M over R by RM . The R -homological dimension of M is defined by

$$\mathrm{hd}_R M = \mathrm{wd} R$$

and, similarly, the R -cohomological dimension of M is defined by

$$\mathrm{cd}_R M = \mathrm{pd} R$$

where R is considered as a trivial RM -module (i.e. $x \cdot r = r$ for $x \in M, r \in R$) and where wd, pd denote weak and projective dimension respectively. Clearly $\mathrm{hd}_R M < \mathrm{cd}_R M$ and the equality holds when RM is noetherian (e.g. when R is noetherian and M is finitely generated abelian). If there is a ring homomorphism $S \rightarrow R$ then $\mathrm{cd}_R M < \mathrm{cd}_S M$ and $\mathrm{hd}_R M < \mathrm{hd}_S M$. In particular, $\mathrm{cd}_R M < \mathrm{cd} M$ and $\mathrm{hd}_R M < \mathrm{hd} M$ where we omit the subscript when $R = \mathbb{Z}$. Recall that a group M is said to have R -torsion if there exists an element of M whose order is not invertible in R . In this case $\mathrm{cd}_R M = \infty$. If M is an $\aleph_n$ -generated abelian group ($n \geq -1$) with no R -torsion, then $\mathrm{hd}_R M = \mathrm{rank} M$ and $\mathrm{cd}_R M = n + 1 + \mathrm{rank} M$ (see, for example, [1], [4] and [10]). In this paper we prove the following.

THEOREM A. *If M is an abelian monoid then $\mathrm{hd}_R M = \mathrm{hd}_R \hat{M}$.*

THEOREM B. *If M is an abelian monoid which is either cancellative or finitely generated, then $\mathrm{cd}_R M = \mathrm{cd}_R \hat{M}$.*

Here $\hat{M}$ denotes the group reflection of M . An example of Nico [8] shows that Theorem B does not hold for arbitrary abelian monoids.

1. Homological dimension of an abelian monoid. The *group reflection* $\hat{M}$ of a monoid M is defined to be the image of M under the left adjoint of the forgetful functor from Groups to Monoids. It is not hard to show that $\hat{M}$ is isomorphic to the quotient of the free group of M by the normal subgroup generated by elements

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of form xyz^{-1} where $xy = z$ and $x, y, z \in M$. There exists a monoid homomorphism $u: M \rightarrow \hat{M}$ taking x to its coset. Note that each monoid homomorphism $M \rightarrow G$ into a group lifts (through u) to a unique group homomorphism $\hat{M} \rightarrow G$. In case M is abelian the group ring $R\hat{M}$ is simply the localization of RM at the central multiplicative set M . If, in addition, M is cancellative then $\hat{M}$ is the group of quotients of M [3, §1.10] and u is injective. In case M is a group we have $\hat{M} = M$.

PROOF OF THEOREM A. Since $R\hat{M}$ is a localization of RM at a central multiplicative set we have that $\text{hd}_R M \geq \text{hd}_R \hat{M}$. To prove the reverse inequality first assume that $R = K$ is commutative. Consider the exact sequence of KM -modules

$$0 \rightarrow I \rightarrow KM \xrightarrow{\varepsilon} K \rightarrow 0$$

where ε sums coefficients. The augmentation ideal I is generated by elements of the form $1 - m$, $m \in M$. Let p be a prime ideal of KM . If $I \not\subseteq p$ then the localization I_p of I at p is simply $(KM)_p$ and so $K_p = 0$. If $I \subseteq p$ then $M \subseteq KM - p$ since otherwise there exists $m \in M \cap p$ and it implies that $1 = m + (1 - m) \in p$. In particular, the elements of M are inverted whenever we localize at prime ideals p such that $K_p \neq 0$. Hence, localizing first at M and then at p is the same as localizing at p . Therefore,

$$\text{hd}_K M = \text{wd}_{KM} K = \sup_p \text{wd}_{KM_p} K \leq \text{wd}_{K\hat{M}} K = \text{hd}_K \hat{M}.$$

In the general case let C denote the center of R . Then $\text{hd}_R M \leq \text{hd}_C M$. On the other hand Swan [9, Proposition 3.3] proved that there exists a prime field F with $\text{hd}_F M \leq \text{hd}_R M$ for any monoid M such that if M is a group with no R -torsion then it has no F -torsion. Suppose $\hat{M}$ has no R -torsion. Then it has no C -torsion nor F -torsion and, therefore, $\text{hd}_C M = \text{hd}_C \hat{M} = \text{rank } \hat{M} = \text{hd}_F \hat{M} = \text{hd}_F M$. Thus, $\text{hd}_R M = \text{hd}_R \hat{M}$. If $\hat{M}$ has R -torsion then $\text{hd}_R \hat{M} = \infty$. Hence, using the inequality proved in the beginning, we have that $\text{hd}_R M \geq \text{hd}_R \hat{M} = \infty$.

COROLLARY 1.1. *If M is finitely generated abelian then $\text{cd}_R M = \text{cd}_R \hat{M}$.*

PROOF. This follows from Theorem A if R is noetherian. Note that $\text{cd}_R M \geq \text{cd}_R \hat{M}$ for any ring R . Clearly the equality holds if $\hat{M}$ has R -torsion. Suppose $\hat{M}$ has no R -torsion. Then $\text{cd}_R M \leq \text{cd } M = \text{cd } \hat{M} = \text{cd}_R \hat{M}$.

REMARK. Corollary 1.1 is no longer true if we remove the finite generation condition on M . For if $M = \{a_i | i \in \mathbb{N}\}$ where $a_i a_j = a_k$ where $k = \max(i, j)$ then $\hat{M} = 1$ and $\text{cd } M \geq 1$ (see, for example, [6] or [8]). More generally $\times_1^n M$ has $\mathbb{Z}$ -cohomological dimension at least n , but its group reflection is also trivial.

2. Cohomological dimension of an abelian monoid. Throughout this section, M will be an (additive) abelian cancellative monoid with group reflection G . Then $u: M \rightarrow G$ is injective and we shall regard M as a submonoid of G . The torsion subgroup of G will be denoted by tG .

LEMMA 2.1. *Suppose a, b are nonnegative integers, not both zero, such that $\text{gcd}(a, b) = 1$. Then the submonoid of the additive group $\mathbb{Z}$ of integers generated by a, b contains all sufficiently large positive integers.*

PROOF. Since $\gcd(a, b) = 1$ there exists $s, t \in \mathbb{Z}$ with $sa + tb = 1$. Note that s and t cannot both be nonpositive. Without loss of generality let $t > 0$. Then by the Euclidean algorithm every integer larger than $a(a-1)|s|$ is of the form $am + k$ where $m > (a-1)|s|$ and $1 \leq k < a$. Since $am + k = am + k(sa + tb) = a(m - ks) + ktb$ is an element of the submonoid, the result follows.

COROLLARY 2.2. *If $G \cong \mathbb{Z}$ then M is finitely generated.*

PROOF. Suppose M contains a positive and a negative integer. Let s and t be the smallest positive and largest negative integer contained in M respectively. Then $t = -s$, for otherwise $s + t$ is either positive which contradicts the minimality of s or it is negative which contradicts the maximality of t . We will prove that $s = 1$ and, therefore, show that $M = G$. Every positive integer of M is of the form $x = gs + r$ where $0 \leq r < s$. Hence $r = x - gs = x + gt \in M$ contradicting the minimality of s except when $r = 0$. Therefore, s divides all positive integers in M . Similarly it divides all the negative integers as well. Thus, M is generated by s and $-s$ and so is a subgroup of G which is a contradiction unless $s = 1$ since G is the group reflection of M . Therefore, M contains either only positive or only negative integers. By symmetry we assume the former. In this case there exists $a, b \in M$ with $a - b = 1$. Hence, $\gcd(a, b) = 1$. By Lemma 2.1 the submonoid generated by a, b contains all sufficiently large positive integers. Therefore, M is finitely generated.

LEMMA 2.3. *Suppose $w \in \mathbb{Z}^m$, $m > 1$, such that two of its coordinates are relatively prime. Then $\mathbb{Z}^m / \langle w \rangle \cong \mathbb{Z}^{m-1}$.*

PROOF. Clearly $\mathbb{Z}^m / \langle w \rangle$ has rank $m - 1$ and hence it is enough to show that it is torsion-free. Suppose $y = (y_i) \in \mathbb{Z}^m$ such that $sy \in \langle w \rangle$ for some $s \in \mathbb{N}$. Then $sy = tw$ for some $t \in \mathbb{Z}$. Without loss of generality we may assume that $\gcd(s, t) = 1$. Therefore, $s|w_i$ for all i . This implies that $s = 1$ by the assumption on w . Hence $y = tw \in \langle w \rangle$.

LEMMA 2.4. *Suppose $G \cong \mathbb{Z}$, $G \neq 0$, and it is finitely generated with no R -torsion. Then there exists $x \in M$ such that $G / \langle x \rangle$ has no R -torsion and $\text{rank } G = 1 + \text{rank } G / \langle x \rangle$. In particular, $\text{cd}_R G / \langle x \rangle$ is finite.*

PROOF. Suppose $\text{rank } G = 1$. Then $G \cong \mathbb{Z} \oplus tG$ where $tG \neq 0$. There exist two elements of M whose difference is $(1, 0)$. Let these elements be (a, b) and $(a + 1, b)$ where $a \in \mathbb{Z}$ and $b \in tG$. By symmetry we may assume that $a > 0$. Since $tG \neq 0$ there exists an element of tG with finite order $n > 1$. Since G has no R -torsion n is invertible in R . Using Lemma 2.1 we see that if t is sufficiently large then there exists a linear combination $y \in M$ of (a, b) and $(a + 1, b)$ with first coordinate n' . Set $x = ky$ where k is the order of b . (Here we take 1 to be the order of the additive identity 0.) Hence, the second coordinate of x is zero. Therefore, $G / \langle x \rangle \cong \mathbb{Z}_{x_1} \oplus tG$ where x_1 denotes the first coordinate of x . Clearly $\text{rank } G / \langle x \rangle = 0 = \text{rank } G - 1$ and, since x_1 is invertible in R , we have that $G / \langle x \rangle$ has no R -torsion.

Assume that $\text{rank } G = m > 1$. Then $G \cong \mathbb{Z}^m \oplus tG$. We first show that there exists $w \in M$ with the first two coordinates relatively prime. There exists $y, y', z, z' \in M$ with $y - y' = (1, 0, \dots, 0)$ and $z - z' = (0, 1, 0, \dots, 0)$. Hence we may let $y = (a, b, \dots)$, $y' = (a - 1, b, \dots)$, $z = (c, d, \dots)$ and $z' = (c, d - 1, \dots)$. If $b \neq 0$ let $w = (b^2 - 1)y + y' = (b^2a - 1, b^3, \dots)$. If $c \neq 0$ let $w = (c^2 - 1)z + z' = (c^3, c^2d - 1, \dots)$. Otherwise $b = 0 = c$. If $d \neq 0$ then we take $w = z'$. If $d \neq 0$ then we take $w = (d^2 - 1)y + y' + z = (d^2a - 1, d, \dots)$. Now set $x = kw$ where k is the order of the last coordinate of w . (Here we take 1 to be the order of 0.) Let w' and x' denote the images of w and x respectively under the projection of G onto $\mathbb{Z}^m$. Since the last coordinate of x is zero we have $G/\langle x \rangle = (\mathbb{Z}^m \oplus tG)/\langle x \rangle \cong \mathbb{Z}^m/\langle x' \rangle \oplus tG$, $\mathbb{Z}^m/\langle x' \rangle \cong \mathbb{Z}^m/\langle w' \rangle \oplus \langle w' \rangle/\langle x' \rangle$. Note $\mathbb{Z}^m/\langle w' \rangle \cong \mathbb{Z}^{m-1}$ by Lemma 2.3 and $\langle w' \rangle/\langle x' \rangle \cong \mathbb{Z}_k$. In any case $\text{rank } G = 1 + \text{rank } G/\langle x \rangle$ and $G/\langle x \rangle$ has no R -torsion since k is invertible in R and G has no R -torsion.

LEMMA 2.5 (KAPLANSKY [5, PART III, THEOREM 3]). *Suppose R is a ring with identity and x is central in R . If x is not a zero divisor and $R^* = R/\langle x \rangle$ then for each R^* -module $A \neq 0$ with $\text{pd}_{R^*} A = n$ we have that $\text{pd}_R A = 1 + n$.*

PROPOSITION 2.6. *If G is finitely generated then $\text{cd}_R M = \text{cd}_R G$.*

PROOF. Since $\text{cd}_R M \geq \text{cd}_R G$ always, we need only to consider the case when $\text{cd}_R G$ is finite. Hence, G has no R -torsion and finite rank. If $\text{rank } G = 0$ then $G = tG$ and so M , as a submonoid of a torsion group, is a group and hence is equal to G . In particular, $\text{cd}_R M = \text{cd}_R G$. Assume $\text{rank } G > 0$. If $G \cong \mathbb{Z}$ then this follows from Corollaries 1.1 and 2.2. Therefore, we may assume $G \not\cong \mathbb{Z}$. By Lemma 2.4 there exists $x \in M$ such that $\text{rank } G = 1 + \text{rank } G/\langle x \rangle$ and $\text{cd}_R G/\langle x \rangle < \infty$. Let $\bar{M}$ denote the image of M under the canonical map $G \rightarrow G/\langle x \rangle = \bar{G}$. It is easy to see that $\bar{G}$ is the group reflection of $\bar{M}$. Hence, by the induction assumption, we have $\text{cd}_R \bar{M} = \text{cd}_R \bar{G}$. On the other hand it is not difficult to show that $R\bar{M} \cong RM/\langle x - 1 \rangle$ and $R\bar{G} \cong RG/\langle x - 1 \rangle$. Since $x - 1 \in RM$ is not a zero divisor and is central in RM we obtain from Lemma 2.5 that $\text{cd}_R M = \text{pd}_{RM} R = 1 + \text{pd}_{R\bar{M}} R = 1 + \text{cd}_R \bar{M}$ and, $\text{cd}_R G = 1 + \text{cd}_R \bar{G}$. Thus $\text{cd}_R M = \text{cd}_R G$.

REMARK. It should be pointed out that M may not be finitely generated even when G is finitely generated. For example let $G = \mathbb{Z}^2$ and $M = \langle \{(m, 1) | m \in \mathbb{N}\} \rangle$. Thus Proposition 2.6 is significantly different from Corollary 1.1.

PROPOSITION 2.7. *Suppose $M = \varinjlim_I M_i$ where I is a directed set of cardinality $\aleph_n$ ($n \geq -1$) and M_i 's are monoids (not necessarily abelian). Then*

$$\text{cd}_R M \leq n + 1 + \sup_I \text{cd}_R M_i.$$

PROOF. This follows from Theorem 37.1 of Mitchell [7] together with the observation that (M, R) is the direct limit of (M_i, R) in $\text{Addfun } Ab$.

PROOF OF THEOREM B. By Corollary 1.1 we may assume that M is cancellative. Since $\text{cd}_R M \geq \text{cd}_R G$ we may assume that $\text{cd}_R G$ is finite. Hence, G has no R -torsion and $\text{rank } G = m < \infty$. If G is finitely generated then the result is simply Proposition 2.6. Assume that G is $\aleph_n$ -generated with $n \geq 0$. Then M is also

$\aleph_n$ -generated, and so $M = \varinjlim M_i$ where M_i 's are finitely generated submonoids of M and $|I| = \aleph_n$. By Proposition 2.7, $\text{cd}_R M \leq n + 1 + \sup \text{cd}_R M_i = n + 1 + \sup \text{cd}_R \hat{M}_i \leq n + 1 + \text{rank } G = \text{cd}_R G$, where $\hat{M}_i$ is the group reflection of M_i .

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