THE AUTOMORPHISM GROUP OF A NEAR-RING

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ABSTRACT. As for groups certain automorphisms of a near-ring N may be regarded as inner. The inner automorphisms of N form a normal subgroup of the automorphism group of N .

Let V be a group, Λ the automorphism group of V , and N the near-ring of maps of V into V generated by Λ . It is shown that if Λ is a complete group (see [2, p. 94]), then it frequently happens that the automorphisms of N are all inner. Using this result the automorphism groups of certain near-rings may be found.

Throughout this paper all near-rings will be left distributive, zero-symmetric and have an identity. If N is a near-ring, then an element of N is a unit if it has a left and a right inverse. The units of N form a group $U(N)$ under composition.

If μ in N is such that $(\alpha + \beta)\mu = \alpha\mu + \beta\mu$ for all α and β in N , then we say μ is distributive. The set of all distributive units of N forms a subgroup of $U(N)$ which we denote by $D(N)$.

A subgroup of $D(N)$ that will be significant in what follows is $Z(N)$. We take $Z(N)$ to be all μ in $D(N)$ such that $\mu^{-1}\alpha\mu = \alpha$ for all α in N . Clearly, if λ is in $D(N)$ and μ is in $Z(N)$, then $\mu^{-1}\lambda\mu = \lambda$ and thus $Z(N)$ is contained in the centre of $D(N)$. It follows that $Z(N)$ is a normal subgroup of $D(N)$.

We shall denote the group of all near-ring automorphisms of N by $A(N)$. If μ is in $D(N)$ then the map $(\mu)\delta$ of N into N that takes α in N to $\mu^{-1}\alpha\mu$ is in $A(N)$ (such maps are an instance of semilinear maps which have been studied by Ramakotaiah [1, p. 194]). Thus δ is a map from $D(N)$ into $A(N)$. We denote the image of δ by $I(N)$. As we shall see in what follows there is an analogy between inner automorphisms of a group and the near-ring automorphisms defined above. Accordingly we call an element of $I(N)$ an *inner automorphism* of N .

PROPOSITION 1. *If N is a near-ring then $I(N)$ is a subgroup of $A(N)$ and $I(N) \cong D(N)/Z(N)$.*

The proof of this proposition parallels the corresponding proof in group theory.

Let N be a near-ring. The subgroup $I(N)$ of $A(N)$ will be called the *group of inner automorphisms* of N .

PROPOSITION 2. *Let N be a near-ring. The subgroup $I(N)$ of $A(N)$ is normal.*

PROOF. Let τ be in $A(N)$ and μ in $D(N)$. Since $\mu\tau$ is in $D(N)$ the proposition will follow if it is shown that $\tau^{-1}(\mu)\delta\tau = (\mu\tau)\delta$. Let α be in N . We have

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$$\alpha[\tau^{-1}(\mu)\delta\tau] = [\mu^{-1}(\alpha\tau^{-1})\mu]\tau = (\mu^{-1}\tau)\alpha(\mu\tau)$$

and since $\mu^{-1}\tau = (\mu\tau)^{-1}$ it follows that

$$\alpha[\tau^{-1}(\mu)\delta\tau] = \alpha[(\mu\tau)\delta].$$

Thus $\tau^{-1}(\mu)\delta\tau = (\mu\tau)\delta$ and the proposition holds.

In Theorem 4 we shall be using Proposition 2 to compute the automorphism group of certain near-rings. We first require a further proposition.

PROPOSITION 3. *Let N be a near-ring, μ an element of $D(N)$ and τ an element of $A(N)$. Assume $Z(N) = \{1\}$. It follows that $(\mu)\delta\tau = \tau(\mu)\delta$ if, and only if, $\mu\tau = \mu$.*

PROOF. In the proof of Proposition 2 we saw that $\tau^{-1}(\mu)\delta\tau = (\mu\tau)\delta$. Thus if $\mu\tau = \mu$ it follows that $(\mu)\delta\tau = \tau(\mu)\delta$.

Assume $Z(N) = \{1\}$. If $\tau^{-1}(\mu)\delta\tau = (\mu)\delta$ then $(\mu\tau)\delta = (\mu)\delta$ and $(\mu\tau)^{-1}\alpha(\mu\tau) = \mu^{-1}\alpha\mu$ for all α in N . Thus

$$[(\mu\tau)\mu^{-1}]^{-1}\alpha(\mu\tau)\mu^{-1} = \alpha$$

for all α in N and, since $(\mu\tau)\mu^{-1}$ is in $D(N)$ and $Z(N) = \{1\}$ it follows that $(\mu\tau)\mu^{-1} = 1$. The proposition now follows.

One class of groups of importance to group theory are the complete groups. A group is complete if it has a trivial centre and all group automorphisms are inner (see [2, p. 94]). By analogy with groups we call a near-ring N *complete* if $I(N) = A(N)$ and $Z(N) = \{1\}$.

THEOREM 4. *Let V be an additive group (not necessarily abelian) and let Λ be the group of automorphisms of V . Let N be the near-ring of maps of V into V generated by Λ . If Λ is a complete group and if there exists v in V such that $vN = V$ then N is a complete near-ring and $A(N) \cong \Lambda$.*

PROOF. Obviously $\Lambda \subseteq D(N)$. Take μ in $D(N)$ and let v be as in the statement of the theorem. If u and w are in V then $u = v\alpha$ and $w = v\beta$ where α and β are in N . Thus

$$\begin{aligned}(u + w)\mu &= (v\alpha + v\beta)\mu = v(\alpha + \beta)\mu \\ &= v\delta\mu + v\beta\mu = u\mu + w\mu\end{aligned}$$

and we conclude that the map μ of V into V is a group homomorphism. Also $N\mu = N$ as μ is a unit and $vN\mu = vN$. Thus μ is onto. If $v\alpha\mu = 0$ for some α in N then $v\alpha = v\alpha\mu\mu^{-1} = 0$ and it follows that μ is in Λ . Hence $\Lambda = D(N)$.

Now $I(N) \cong D(N)/Z(N)$ by 1. Since $\Lambda = D(N)$, Λ is a complete group, and $Z(N)$ is contained in the centre of $D(N)$, it follows that $I(N) \cong \Lambda$.

By 2, $I(N)$ is a normal subgroup of $A(N)$ and by [2, p. 94], $I(N)$ is a direct factor of $A(N)$. Thus $A(N) = I(N) \otimes U$ where U is a subgroup of $A(N)$. The theorem will follow if we show that U is trivial.

Let τ be in U . We have $(\mu)\delta\tau = \tau(\mu)\delta$ for all μ in $D(N)$ ($= \Lambda$). Since $Z(N)$ is contained in the centre of $D(N)$ which is trivial it follows from 3 that $\mu\tau = \mu$ for all μ in $D(N)$ ($= \Lambda$). Now if α is in N then

$$\alpha = \mu_1 + \cdots + \mu_k$$

where $-\mu_i$ or μ_i , $i = 1, \dots, k$, are in Λ . Hence

$$\alpha\tau = \mu_1\tau + \dots + \mu_k\tau$$

and from the above $\mu_i\tau = \mu_i$ for $i = 1, \dots, k$. It follows that $\alpha\tau = \alpha$ for all α in N and τ is the identity. Thus U is trivial and the theorem holds.

We state as a corollary a special case of Theorem 4.

COROLLARY. *Let V be an additive group (not necessarily abelian). Let N be the near-ring generated by the inner automorphisms of V and suppose there exists v in V such that $vN = V$. If V is complete then N is a complete near-ring and $A(N) \cong V$.*

EXAMPLES. Let S_n , $n \geq 3$, $n \neq 6$, be the symmetric group on n letters. Suppose N is the near-ring generated by the inner automorphisms of S_n . Since S_n is complete (see [2, p. 96]) and $S_n = vN$ for any v not contained in A_n (the alternating subgroup) it follows that $A(N) \cong S_n$ and N is complete.

The above example is one of many. The holomorph V of a finite cyclic group of odd order is complete (see [2, p. 96]) and there exists a v in V as in the above corollary. Thus for such a group the near-ring N generated by the inner automorphisms of V has the property that it is complete and $A(N) \cong V$.

Let V be a group and $M_0(V)$ the near-ring of all zero-fixing maps of V . Ramakotaiah has shown (see [1, p. 196]) that $M_0(V)$ is complete and $A(M_0(V))$ is isomorphic to the automorphism group of V .

One question, a group theory question, that arises from the above is which complete groups satisfy the conditions of the above corollary i.e. which complete groups are the normal closure of a single element.

Another question is whether or not this condition is needed in Theorem 4 and the above corollary.

REFERENCES

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