WEAK CONTINUITY OF INVARIANT MEASURES FOR A CLASS OF PIECEWISE MONOTONIC TRANSFORMATIONS

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ABSTRACT. Let $\tau_a: [0, 1] \rightarrow [0, 1]$, $0 < a < \frac{1}{2}$, denote the class of transformations defined by

$$\tau_a(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2}, \\ (2-a) - 2(1-a)x, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

For $a^* < a < \frac{1}{2}$, the transformation τ_a admits unique absolutely continuous invariant measures μ_a whose density function $f_a(x)$ assumes the value 0 on an interval which expands as $a \uparrow \frac{1}{2}$. Furthermore, the measures μ_a converge weakly to the measure $\mu_{1/2} = \frac{1}{2}\delta_{1/2} + \frac{1}{2}\delta_1$ as $a \uparrow \frac{1}{2}$, where $\mu_{1/2}$ is an invariant measure under τ_a , and δ_y denotes the Dirac measure at the point y .

1. Introduction. Let $\tau: [0, 1] \rightarrow [0, 1]$ be a piecewise C^2 transformation satisfying $\inf_x |d\tau/dx| > 1$, where the slope exists. Lasota and Yorke [2] established the existence of absolutely continuous invariant measures for this class of transformations.

The question of continuity of the invariant measures with respect to the transformation itself is considered in [1]. Under certain restrictions, one of which is that the slopes of all segments of τ be bounded away from 1, Kowalski [1, Theorem 3] gives necessary and sufficient conditions that guarantee: $\tau_n \rightarrow \tau$ implies $f_n \rightarrow f$, where f_n, f are the densities of the measures μ_n, μ invariant under τ_n, τ , respectively. The convergence of τ_n and f_n are both in the L_1 -norm on $[0, 1]$.

The object of this note is to give an example of $\tau_n \rightarrow \tau$ such that $\mu_n \Rightarrow \mu$, where τ has slope 1 on an interval, μ is a point measure invariant under τ , and $\Rightarrow$ means weak convergence.

2. Properties of $\{\tau_a\}$. Let $\tau_a: [0, 1] \rightarrow [0, 1]$ denote a class of transformations labelled by a , $0 < a < \frac{1}{2}$, and defined by

$$\tau_a(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2}, \\ (2-a) - 2(1-a)x, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

For $0 < a < \frac{1}{2}$, it follows from [3] that τ_a admits a unique absolutely continuous invariant measure μ_a , whose density with respect to the Lebesgue measure is denoted by f_a .

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LEMMA 1. Let $a^* = (7 - \sqrt{17})/8$. For $a^* < a < \frac{1}{2}$, $f_a(x) = 0$ when $4a^2 - 5a + 2 < x < 2a$.

PROOF. Let $0 < a < \frac{1}{2}$. Since no orbit $\{\tau_a^i(x)\}$, $x \in (0, 1]$ can return to $(0, a)$ after it has left it, the support of $f_a \subset [a, 1]$. Let $\varepsilon > 0$, $\delta > 0$ and define $I_a = [a, a + \varepsilon]$ and $J_a = [1 - \delta, 1]$. Assume further that $\frac{1}{2} < a + \varepsilon < 1 - \delta$ and $\tau_a(a) < \tau_a(a + \varepsilon)$. Then $\tau_a(I_a) = [2a, 1]$ and $\tau_a(J_a) = [a, (2 - a) - 2(1 - a)(1 - \delta)]$. We now show that there exist ε and δ such that $I_a \cup J_a$ is τ_a -invariant. (A set X is τ_a -invariant if $\tau_a(X) = X$ a.e.)

We impose two conditions: $\tau_a(I_a) = J_a$ and $\tau_a(J_a) = I_a$. The first requires that $[2a, 1] = [1 - \delta, 1]$. Thus $\delta = 1 - 2a$. The second condition requires that $[a, (2 - a) - 2(1 - a)(1 - \delta)] = [a, a + \varepsilon]$. Thus $\varepsilon = 2(1 - a)(1 - 2a)$. For these values of ε and δ , $I_a \cup J_a$ is τ_a -invariant. Since the support of the unique invariant measure μ_a is τ_a -invariant, the support of f_a is contained in $I_a \cup J_a$.

The set I_a will meet the set J_a when $a + \varepsilon = 1 - \delta$. This occurs when $a + 2(1 - a)(1 - 2a) = 2a$. The roots of this quadratic are $(7 \pm \sqrt{17})/8$. Only $a^* = (7 - \sqrt{17})/8$ falls into the interval $[0, 1]$. Thus $I_{a^*} \cap J_{a^*} = \{2a^*\}$.

For $a^* < a < \frac{1}{2}$, $\varepsilon_a \equiv 2(1 - a)(1 - 2a) < \varepsilon_{a^*}$ and $1 - \delta_a \equiv 2a > 1 - \delta_{a^*}$. Clearly, $I_a \subset I_{a^*}$ and $J_a \subset J_{a^*}$, where the containment is strict. Since the support of f_a is contained in $I_a \cup J_a = [a, 4a^2 - 5a + 2] \cup [2a, 1]$, the conclusion follows. Q.E.D.

From the definition of the Frobenius-Perron Operator [2], it can be easily shown that f_a is the solution of the functional equation

$$f_a(x) = \begin{cases} 0, & 0 < x < a, \\ \frac{1}{2(1-a)} f_a\left(\frac{1}{2} + \frac{1-x}{2(1-a)}\right), & a < x < 2a, \\ \frac{1}{2} f_a\left(\frac{x}{2}\right) + \frac{1}{2(1-a)} f_a\left(\frac{1}{2} + \frac{1-x}{2(1-a)}\right), & 2a < x < 1, \end{cases} \quad (1)$$

such that

$$\int_0^1 f_a(x) dx = 1. \quad (2)$$

Equation (1) provides an alternative proof of Lemma 1, which goes as follows. It can be shown that if $4a^2 - 5a + 2 < x < 2a$, then

$$4a^2 - 5a + 2 < \frac{1}{2} + (1 - x)/2(1 - a) < 2a$$

where $a^* < a < \frac{1}{2}$. For $0 < a < \frac{1}{2}$, $0 < 1/2(1 - a) < 1$, and from [2] it follows that f_a is bounded. Starting with any point x in the interval $(4a^2 - 5a + 2, 2a)$, repeated iterations by f_a show that for any $\varepsilon > 0$, $f_a(x) < \varepsilon$. Hence, $f_a(x) = 0$. When $a = a^*$, $f_{a^*}(2a^*) = 0$.

Returning to equation (1), let us define $F_a(x) = \int_0^x f_a(t) dt$. From (1) and (2), it follows that

$$F_a(x) = \begin{cases} 0, & 0 \leq x \leq a, \\ 1 - F_a\left(\frac{1}{2} + \frac{1-x}{2(1-a)}\right), & a < x \leq 2a, \\ 1 + F_a\left(\frac{x}{2}\right) - F_a\left(\frac{1}{2} + \frac{1-x}{2(1-a)}\right), & 2a < x \leq 1. \end{cases} \quad (3)$$

Define x_a to be the number satisfying $x_a = \frac{1}{2} + (1 - x_a)/2(1 - a)$. Thus $x_a = (2 - a)/(3 - 2a)$. Clearly $x_a < 2a$ for these values of a . Since $a + \varepsilon_a = 4a^2 - 5a + 2 < (2 - a)/(3 - 2a)$ for $(7 - \sqrt{17})/8 < a < \frac{1}{2}$, we get $x_a \notin I_a \cup J_a$. Hence $f_a(x_a) = 0$. Now, from (3), we have $F_a(x_a) = 1 - F_a(x_a)$. Thus, $F_a(x_a) = \frac{1}{2}$. Q.E.D.

For the transformation

$$\tau_{1/2}(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 3/2 - x, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

there are an uncountable number of point measures invariant under $\tau_{1/2}$. Indeed any measure supported on the two points $\{x, 1.5 - x\}$, where $\frac{1}{2} \leq x \leq 1$, is an invariant measure. In particular $\{\frac{1}{2}, 1\}$ is the support of an invariant measure. Since the orbit going through these two points is a 2-cycle, they must have equal weight. Hence, let $\mu_{1/2} = \frac{1}{2}\delta_{1/2} + \frac{1}{2}\delta_1$. For this invariant measure, we prove

THEOREM 1. $\mu_a \Rightarrow \mu_{1/2}$ as $a \uparrow \frac{1}{2}$.

PROOF. From Lemma 1, it follows that $I_a \downarrow \{\frac{1}{2}\}$ and $J_a \uparrow \{1\}$ as $a \uparrow \frac{1}{2}$. Thus, the supports of μ_a approach the set $\{\frac{1}{2}, 1\}$ as $a \uparrow \frac{1}{2}$. Let $F_{1/2}(x)$ be the distribution function induced by $\mu_{1/2}$, i.e.,

$$F_{1/2}(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} \leq x < 1, \\ 1, & x = 1. \end{cases}$$

Using $F_a(x_a) = \frac{1}{2}$, it follows that $F_a(x) \rightarrow F_{1/2}(x)$ pointwise for all x except $x = \frac{1}{2}$ and 1 as $a \uparrow \frac{1}{2}$. This is equivalent to the assertion of the theorem. Q.E.D.

Final note. It would be of interest if one could find a class of transformations admitting slope 0 (for example, $\tau_a(x) = ax(1 - x)$, $0 \leq a < 1$), whose invariant measures are weakly continuous with respect to a , at least at some a . This would be an important step in justifying the use of such discrete models in practice. Numerical results indicate that there is weak continuity of invariant measures from the left for $\tau_4(x) = 4x(1 - x)$.

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