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TWO NEW EXTREMAL PROPERTIES OF THE KOEBE-FUNCTION[1]

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ABSTRACT. Using essentially Löwner's method the extremality of the Koebe-functions with respect to two coefficient problems for inverses of univalent functions is proved.

Let $D = \{z| \, |z| < 1\}$ and $S = \{f|f$ regular and univalent in $D, f(0) = f'(0) - 1 = 0\}$. K. Löwner [4] proved: If $F(w) = w + \sum_{n=2}^{\infty} A_n w^n$ is the inverse of a function in S, then

$$|A_n| \leq (2n)!/n!\,(n+1)!$$

with equality only for the inverses of the Koebe-functions $k_\sigma(z) = z(1 + \sigma z)^{-2}$, $|\sigma| = 1$.

In this note we shall prove similar results for the functions

$$\ln F'(w), \qquad \Delta(F(w), w) := (F''/F')' - \tfrac{1}{2}(F''/F')^2.$$

This work was stimulated by a conjecture of the first author (see [2] and [3]) and the preprint [6] of a lecture given by G. Schober at the Durham Conference on Aspects of Contemporary Complex Analysis in 1979.

THEOREM. *Let F be the inverse of a function in S,* $K_1(w) = k_1^{-1}(w)$,

$$\ln F'(w) = \sum_{n=1}^{\infty} B_n w^n, \qquad \ln K_1'(w) = \sum_{n=1}^{\infty} b_n w^n,$$

$$\Delta(F(w), w) = \sum_{n=0}^{\infty} C_n w^n, \qquad \Delta(K_1(w), w) = \sum_{n=0}^{\infty} c_n w^n.$$

Then $|B_n| \leq b_n$ *for* $n \in \mathbf{N}$ *and* $|C_n| \leq c_n$ *for* $n \in \mathbf{N} \cup \{0\}$. *Equality for* $n \in \mathbf{N}$ *occurs only for the functions* $K_\sigma(w) = k_\sigma^{-1}(w)$, $|\sigma| = 1$.

REMARKS. In the case of the Schwarzian derivative $\Delta(K_1(w), w)$ we have the simple representation $c_n = 4^n 6(n+1)$, $n \in \mathbf{N} \cup \{0\}$ (see [3]). The first part of the theorem implies Löwner's theorem since each A_n is a polynomial with positive coefficients in the B_n.[2]

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[2]This was pointed out by the referee.

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PROOF. The proof follows the same line as the famous proof of Löwner's result (see f.i. [1], [4], [6]). So we need only give here the crucial steps.

If $f \in S$, f can be embedded into a subordination chain. It results that F, the inverse of f, has a representation

$$F(w) = \lim_{t\to\infty} \Phi(e^{-t}w, t), \qquad \partial\Phi(w, t)/\partial t = w(\partial\Phi(w, t)/\partial w)p(w, t) \tag{1}$$

with

$$p(w, t) = 1 + \sum_{n=1}^{\infty} p_n(t)w^n, \quad \operatorname{Re} p(w, t) > 0 \quad \text{for } w \in D, t > 0, \Phi(w, 0) = w. \tag{2}$$

(For details see [5].)

Using (1) and (2) and setting

$$L(w, t) := \ln \frac{\partial\Phi(w, t)}{\partial w} = \sum_{n=0}^{\infty} B_n(t)w^n,$$

$$\Delta(w, t) := \Delta(\Phi(w, t), w) = \sum_{n=0}^{\infty} C_n(t)w^n,$$

we get

$$\partial L/\partial t = (\partial L/\partial w)wp + (\partial/\partial w)(wp),$$

$$\partial\Delta/\partial t = (\partial\Delta/\partial w)wp + 2\Delta(\partial/\partial w)(wp) + (\partial^3/\partial w^3)(wp),$$

$$B_0(t) = t, \qquad B_n(t) = \int_0^t e^{n(t-\tau)}\left(\sum_{j=1}^{n-1} jB_j(\tau)p_{n-j}(\tau) + (n+1)p_n(\tau)\right)d\tau, \qquad n \in \mathbf{N}, \tag{3}$$

$$C_n(t) = \int_0^t e^{(n+2)(t-\tau)}\left(\sum_{j=0}^{n-1} C_j(\tau)p_{n-j}(\tau)(2n-j+2) + \frac{(n+3)!}{n!}p_{n+2}(\tau)\right)d\tau, \qquad n \in \mathbf{N} \cup \{0\}, \tag{4}$$

$$B_n = \lim_{t\to\infty} e^{-nt}B_n(t), \qquad n \in \mathbf{N}, \tag{5}$$

$$C_n = \lim_{t\to\infty} e^{-(n+2)t}C_n(t), \qquad n \in \mathbf{N} \cup \{0\}. \tag{6}$$

(3) and (4) show that Re $B_n(t)$, resp. Re $C_n(t)$ is maximal for fixed t if and only if we choose $B_j(\tau)$, $j = 1, \ldots, n-1$, resp. $C_j(\tau)$, $j = 0, \ldots, n-1$, $\tau \in [0, t]$ real and maximal and any $p_j(\tau)$ involved in (3), resp. (4), equal to the constant 2. As a consequence of (5) and (6) we get that Max Re B_n, resp. Max Re C_n, $n \in \mathbf{N}$, is attained if and only if $p_1(t) \equiv 2$ which means $p(w, t) = (1 + w)/(1 - w)$. Now the assertion of the theorem for $n \in \mathbf{N}$ follows from the fact that the problems of finding the maximum of the real part and the maximum of the modulus for the given coefficients are equivalent (up to a rotation).

The equality $C_0 = -f^{(3)}(0) + \frac{3}{2}(f''(0))^2$ shows that the remaining case is a classical inequality.

References

1. W. K. Hayman, *Multivalent functions*, Cambridge Univ. Press, London and New York, 1958.

2. R. Klouth, *Abschätzungen für verallgemeinerte Schwarzsche Derivierte und gewisser Verallgemeinerungen*, Dissertation, Bonn. Math. Schr. Nr. 82, 1976.

3. ______, *Abschätzungen verallgemeinerter Schwarzscher Derivierter für schlichte holomorphe Funktionen im Einheitskreis*, Preprint des SFB 40 Theoretische Mathematik, Bonn, 1979.

4. K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*. I, Math. Ann. **89** (1923), 103–121.

5. Chr. Pommerenke, *Univalent functions*, Vandenhoeck und Ruprecht, Göttingen, 1975.

6. G. Schober, *Coefficient estimates for inverses of Schlicht functions* (preprint).

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