NONNORMAL DIRICHLET QUOTIENTS AND NONNORMAL BLASCHKE QUOTIENTS

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ABSTRACT. There exists a nonnormal meromorphic function f_1/f_2 in $U = \{|z| < 1\}$, where f_1 and f_2 both are holomorphic functions with finite Dirichlet integrals in U . For each $0 < \alpha < 1$, there exists a nonnormal meromorphic function B_1/B_2 in U , where B_1 and B_2 both are Blaschke products with finite α -weighted Dirichlet integrals in U .

1. Introduction. Let D be the family of functions f holomorphic in $U = \{|z| < 1\}$ with finite Dirichlet integral

$$\iint_U |f'(z)|^2 dx dy < \infty \quad (z = x + iy).$$

Then, a meromorphic function g of the form $g = f_1/f_2$, where $f_j \in D$, $j = 1, 2$, is called a Dirichlet quotient in U . Note that each $f \in D$ is normal in U in the sense of O. Lehto and K. I. Virtanen [5]. The question then arises as to whether the quotient of two such normal functions is again normal in U .

THEOREM 1. *There exists a nonnormal Dirichlet quotient in U .*

Theorem 1 was claimed by N. Yanagihara [8]. However, as is pointed out by D. C. Rung in *Mathematical Reviews* 36 (1968), #1622, Yanagihara's "proof" is unfortunately a mixture of confusion and inaccuracy. We present a new proof which we believe is more comprehensible than Yanagihara's.

Let D_α ($0 < \alpha < 1$) be the family of functions f holomorphic in U with finite weighted Dirichlet integral $\iint_U (1 - |z|)^\alpha |f'(z)|^2 dx dy < \infty$. A quotient B_1/B_2 of two infinite Blaschke products B_1 and B_2 with no common zero is called a Blaschke quotient in U . Our next result is

THEOREM 2. *For each $0 < \alpha < 1$, there exists a nonnormal Blaschke quotient B_1/B_2 in U such that $B_j \in D_\alpha$, $j = 1, 2$.*

It is easy merely to construct a nonnormal Blaschke quotient. The existence of the quotient with an additional property is noted by J. A. Cima [2, Theorem 4]; see the remark at the end of §3.

Here we note that $D = D_0 \subset D_\alpha \subset D_\beta \subset D_1 = H^2$ for $0 < \alpha < \beta < 1$, where H^2 is the Hardy class in U [3]. Furthermore, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in U$, is of D_α ($0 < \alpha < 1$) if and only if

$$\sum_{n=0}^{\infty} n^{1-\alpha} |a_n|^2 < \infty;$$

Received by the editors September 13, 1979 and, in revised form, February 7, 1980.
AMS (MOS) subject classifications (1970). Primary 30A74, 30A76.

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0002-9939/80/0000-0614/\$01.75

see, e.g., [4, pp. 8–9]. For the proof of $D_1 = H^2$, see [9, Remark (a), p. 208]. Thus, $f \in D_1$ if and only if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

2. Proof of Theorem 1. Suppose that $\{z_n\}_{n=1}^{\infty}$ is a sequence of points in U , converging to 1 along the chord X of the unit circle connecting 1 and i . Suppose further that

$$\sum_{n=1}^{\infty} \left(\log \frac{1}{1 - |z_n|^2} \right)^{-1} < \infty.$$

Then it follows from [7, Theorem 1, p. 220] (see also the last sentence in [7, p. 220]) that there exists a function $g \in D$ whose zeros are exactly 0 and z_n , $n = 1, 2, \dots$. The function we wish to examine is the quotient $f = f_1/f_2$, where

$$f_1(z) = (z - 1)g(z) \quad \text{and} \quad f_2(z) = \overline{g(\bar{z})}, \quad z \in U.$$

To prove that f is a Dirichlet quotient we note first that $f_2 \in D$. To show that $f_1 \in D$ let $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $z \in U$. Then

$$\int \int_U |g(z)|^2 dx dy = \pi \sum_{n=0}^{\infty} (n+1)^{-1} |b_n|^2 < \pi \sum_{n=0}^{\infty} n |b_n|^2 < \infty.$$

Since $|f'_1|^2 \leq 2|g|^2 + 8|g'|^2$, it follows that $f_1 \in D$. To prove the nonnormality of f , we first note that $|f(z)| = |z - 1| \rightarrow 0$ as $z \rightarrow 1$ along the radius $R = \{0 < z < 1\}$ of U at 1. Assume that f is normal in U . It then follows from the result of Lehto and Virtanen [5, Theorem 2] that f must possess zero as the angular limit at 1. This is a contradiction because $f(\bar{z}_n) = \infty$, $n = 1, 2, \dots$.

3. Proof of Theorem 2. We begin with a lemma on a Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|c_n|}{c_n} \frac{c_n - z}{1 - \bar{c}_n z}, \quad z \in U, \quad (3.1)$$

where $0 < |c_n| < 1$ ($n = 1, 2, \dots$) and $\sum (1 - |c_n|) < \infty$.

LEMMA. Suppose that $\sum_{n=1}^{\infty} (1 - |c_n|)^{\beta} < \infty$ for $0 < \beta < 1$. Then B of (3.1) is a member of D_{β} .

In effect, it follows from [6, Theorem 1] that

$$\int \int_U (1 - |z|)^{\beta-1} |B'(z)| dx dy < \infty. \quad (3.2)$$

On the other hand, the Schwarz-Pick lemma yields

$$(1 - |z|)|B'(z)| \leq (1 - |z|^2)|B'(z)| \leq 1 - |B(z)|^2 < 1,$$

whence

$$(1 - |z|)^{\beta} |B'(z)|^2 \leq (1 - |z|)^{\beta-1} |B'(z)|, \quad z \in U. \quad (3.3)$$

Combining (3.2) and (3.3) one observes that $B \in D_{\beta}$.

For the proof of Theorem 2, we first choose a sequence $\{a_n\}_{n=1}^{\infty}$ of points in U converging to 1 along the chord X . We further suppose that $|a_n| = 1 - \gamma^n$, $n = 1, 2, \dots$, where $0 < \gamma < 1$ is a constant. Then

$$\sum_{n=1}^{\infty} (1 - |a_n|)^{\alpha} < \infty. \quad (3.4)$$

Next choose $\lambda > 1/\alpha$ and set $b_n = 1 - (n+1)^{-\lambda}$, $n = 1, 2, \dots$. Consider the Blaschke products

$$B_3(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \quad \text{and} \quad B_4(z) = \prod_{n=1}^{\infty} \frac{b_n - z}{1 - \bar{b}_n z}.$$

The Blaschke quotient we wish to examine is $Q = B_1/B_2$, where

$$B_1(z) = B_3(z)B_4(z) \quad \text{and} \quad B_2(z) = \overline{B_3(\bar{z})}, \quad z \in U.$$

Since $\sum_{n=1}^{\infty} (1 - b_n)^{\alpha} < \infty$, it follows from the lemma, together with (3.4), that both B_1 and B_2 are of class D_{α} . To prove the nonnormality of Q we first prove that $B_4(z) \rightarrow 0$ as $z \rightarrow 1$ along the radius R . Denoting the non-Euclidean distance in U by $\rho(\cdot, \cdot)$, we observe that the sequence

$$\rho(b_n, b_{n+1}) = \frac{1}{2} \log \frac{1 - b_n b_{n+1} + b_{n+1} - b_n}{1 - b_n b_{n+1} - b_{n+1} + b_n}$$

tends to zero as $n \rightarrow \infty$. It then follows from [1, Theorem 2] that the bounded function B_4 has the angular limit zero at 1. By the same reasoning as in the last part of §2, Q is not normal in U .

REMARK. Cima [2, Theorem 4] found a nonnormal Blaschke quotient

$$\left(\prod_{n=1}^{\infty} \frac{|c_n|}{c_n} \frac{c_n - z}{1 - \bar{c}_n z} \right) / \left(\prod_{n=1}^{\infty} \frac{|c'_n|}{c'_n} \frac{c'_n - z}{1 - \bar{c}'_n z} \right)$$

such that $\inf_{j,k \geq 1} \rho(c_j, c'_k) > 0$; in other words, the distance between the zeros $\{c_n\}$ and the poles $\{c'_n\}$ is positive. Our quotient Q also satisfies this property. For, let $\bar{X}$ be the reflexion of X with respect to the real axis. Then the non-Euclidean distance between $\bar{X}$ and the union $X \cup R$ is positive.

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