ON ONESIDED HARMONIC ANALYSIS

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ABSTRACT. It is shown that, in locally compact groups satisfying the onesided version of the Wiener property introduced by Leptin, characters of closed subgroups have always continuous positive definite extensions onto the whole group. We give a quick proof that for such groups the connected component of the identity is the direct product of a compact group and a vector group.

In [4], Leptin considers the following onesided version of Wiener's theorem for a locally compact group G .

- (L) Every closed proper left ideal in $L^1(G)$ is annihilated by nonzero positive linear functionals (or equivalently, every nonzero weak- $*$ -closed left translation-invariant subspace of $L^\infty(G)$ contains nonzero positive definite functions).

By reduction to some special groups Leptin shows that a connected group G satisfies property (L) if and only if G is the direct product of a compact group and a commutative group. In this note we give a short proof of the nontrivial part of that result and of some generalizations to not necessarily connected groups. In particular, if G is almost connected and satisfies property (L), then G is the semidirect product of a vector group V and a compact group K such that K acts on V effectively as a finite group.

Let $C_{00}(G)$ be the continuous functions on G with compact support. For $f \in C_{00}(G)$ let $f|H$ denote its restriction onto the subgroup H and let ${}_xf(y) = f(xy)$, $x, y \in G$. $\langle \varphi, f \rangle = \int_G \varphi(x)f(x) dx$ is the canonical bilinear form on $L^\infty(G) \times L^1(G)$, where dx denotes a fixed left Haar measure on G .

Character always means a continuous homomorphism into the circle.

THEOREM 1. *Let G be a locally compact group. Suppose that every closed proper left ideal in $L^1(G)$ is annihilated by nonzero positive linear functionals. Then characters of closed subgroups of G can always be extended as continuous positive definite functions of G .*

PROOF. Let γ be a character of a closed subgroup H of G and let I^γ be the right ideal in $C_{00}(G)$ defined by $I^\gamma = \{v \in C_{00}(G); \langle \gamma, v * u|H \rangle = 0 \text{ for all } u \in C_{00}(G)\}$.

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As usual, by taking approximate identities for $C_{00}(G)$ with the inductive limit topology, one shows that I^γ is a proper right ideal. Let R be the closure of I^γ in $L^1(G)$ with respect to the $L^1(G)$ -norm. Since for $v, u \in C_{00}(G)$

$$|\langle \gamma, v * u | H \rangle| \leq \|v * u|H\|_{L^1(H)} \leq \|v\|_{L^1(G)} \sup_{x \in G} \int_H |u(xs)| ds$$

it follows that $R \cap C_{00}(G) = I^\gamma$. Hence R is a proper right ideal in $L^1(G)$.

Now let φ be a nonzero continuous positive definite function of G which annihilates the left ideal $L = \{v^*; v \in R\}$. Since $\varphi(x^{-1}) = \overline{\varphi(x)}$, $x \in G$, R also is annihilated by φ , we may assume that $\varphi(e) = 1$. For $t \in H, f \in C_{00}(G)$ we have

$$\begin{aligned} \langle \gamma, {}_t f * u | H \rangle &= \int_H \gamma(s) \int_G f(t^{-1}sy) u(y^{-1}) dy ds \\ &= \int_G u(y^{-1}) \int_H \gamma(ts) f(sy) ds dy = \gamma(t) \langle \gamma, f * u | H \rangle \end{aligned}$$

for all $u \in C_{00}(G)$. Therefore ${}_t f - \gamma(t)f \in R$, hence

$$0 = \langle \varphi, {}_t f \rangle - \langle \varphi, \gamma(t)f \rangle = \int_G \{ \varphi(tx) - \varphi(x)\gamma(t) \} f(x) dx$$

holds for all $f \in C_{00}(G)$.

Since φ is continuous it follows that

$$\varphi(tx) = \varphi(x)\gamma(t) \quad \text{for all } x \in G, t \in H.$$

Setting $x = e$ we get $\varphi|H = \gamma$.

REMARK. Since the extension property as stated in Theorem 1 is inherited by closed subgroups we can use Theorem 1 in [3] to see that in groups satisfying the onesided Wiener property (L) every connected closed subgroup is the direct product of a compact group and a vector group. By analyzing the proof of Theorem 1, however, we can show that the same holds under some weaker conditions and we shall give a proof which is considerably shorter than those given in [3] and [4].

For if φ is any continuous function in $L^\infty(G)$ which annihilates the right ideal R then by the proof of Theorem 1

$$\varphi(tx) = \gamma(t)\varphi(x) \quad \text{holds for } t \in H, x \in G.$$

This equation is certainly satisfied by many functions but if φ is hermitian, i.e. $\varphi(x^{-1}) = \overline{\varphi(x)}$, $x \in G$, then also

$$\varphi(xt) = \overline{\varphi(t^{-1}x^{-1})} = \overline{\gamma(t^{-1})} \overline{\varphi(x^{-1})} = \gamma(t)\varphi(x)$$

holds for $x \in G, t \in H$. Thus we have proved

LEMMA 1. *If every nonzero weak- $*$ -closed left translation-invariant subspace in $L^\infty(G)$ contains continuous functions φ such that $\varphi(x^{-1}) = \overline{\varphi(x)}$, $x \in G$, and $\varphi(e) \neq 0$ then G has the following property.*

- (C) *For every closed subgroup H of G and every character γ of H there is a continuous function φ on G such that $\varphi(e) \neq 0$ and $\varphi(tx) = \varphi(xt) = \gamma(t)\varphi(x)$ for all $t \in H, x \in G$.*

The first statement of the following lemma is obvious.

LEMMA 2. (1) *Property (C) is inherited by closed subgroups and quotients of locally compact groups.*

(2) *If a connected locally compact group G has property (C) then closed abelian normal subgroups are contained in the center.*

PROOF. (Compare [3, Proposition 1 and Lemma 2].) Let G have property (C) and let γ be a character of a closed normal subgroup H of G . Then there is a nonzero continuous function φ of G such that

$$\varphi(x)\gamma(t) = \varphi(xt) = \varphi(xtx^{-1}) = \gamma(xtx^{-1})\varphi(x)$$

for $x \in G$, $t \in H$. Hence if $\varphi(x) \neq 0$, x is contained in the stationary subgroup $F(\gamma) = \{x \in G; \gamma(xtx^{-1}) = \gamma(t) \text{ for all } t \in H\}$. Thus $F(\gamma)$ is an open subgroup.

If H is abelian and not contained in the center of G , then there is a character γ of H such that $F(\gamma) \neq G$; hence G cannot be connected.

Property (C) is rather restrictive in the case of connected groups by Lemma 2 and the following general fact. The arguments used in the proof are essentially the same as those used in [3] to prove Theorem 1.

THEOREM 2. *Let (P) be any property which locally compact groups may have such that the following conditions hold.*

(1) *Property (P) is inherited by quotients and connected closed subgroups.*

(2) *If a connected group G has property (P) then closed abelian normal subgroups of G are contained in the center.*

Then every connected group G satisfying property (P) is the direct product of a compact group and a vector group.

PROOF. (a) Assume first that G is solvable. If G is not abelian, then by condition (2) there are closed normal subgroups H , L of G such that L is contained in the center Z of G , H/L is abelian and $H \not\subseteq Z$. Since G/L has property (P), H/L is contained in the center of G/L . Hence for arbitrary $x \in H$ the closed subgroup generated by x and L is an abelian normal subgroup of G and therefore is contained in Z by condition (2) again. This contradicts $H \not\subseteq Z$.

(b) Let G be semisimple. In order to show that then G is compact we may assume that G is a Lie group [1, p. 54]. Using the Iwasawa decomposition one knows that a noncompact semisimple connected Lie group contains a closed solvable connected subgroup S which is not abelian (see [3, Lemma 3] for more details). By (a), S cannot satisfy (P) and from condition (1) the assertion follows.

(c) *The general case.* By (a) and condition (1), the radical $\text{rad}(G)$ of G is abelian hence contained in Z . By (b), $G/\text{rad}(G)$ is compact thus G/Z is compact. Finally, G is the direct product of a compact group and a vector group by the Freudenthal-Weil Theorem.

Combining Theorem 2 with Lemma 1 and 2 we get

THEOREM 3. *Let G be a locally compact group and let H be a connected closed subgroup which is not the direct product of a compact group and a vector group. Then there is a nonzero weak- $*$ -closed left translation-invariant subspace in $L^\infty(G)$ which contains no continuous functions φ such that $\varphi(x^{-1}) = \overline{\varphi(x)}$, $x \in G$, and $\varphi(e) \neq 0$. In particular, G does not satisfy the onesided Wiener property (L).*

COROLLARY. *Let G be almost connected (i.e. G/G_0 is compact where G_0 is the identity component). If G has property (L) then it is the semidirect product of a vector group V and a compact group K such that K acts on V effectively as a finite group.*

PROOF. By Theorem 3, G_0 is the direct product of a vector group and a compact group. Then G has the structure as stated in the Corollary by [2, Theorem 2.9] (see also [3, Theorem 2]).

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