ON A CRITERION FOR THE UNIVALENCE OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. A sufficient condition for the univalence of holomorphic functions in the unit disc is given in terms of $|f''/f'|$.

Let f be a function nonconstant and holomorphic in $D = \{z \in \mathbb{C}: |z| < 1\}$. Let $\alpha > 0$ and let

$$\|f\|_\alpha = \sup_{z \in D} (1 - |z|^2)^\alpha |f''(z)/f'(z)|.$$

In this paper we shall prove

THEOREM. *If $\alpha \in (0, 1]$ write*

$$K_\alpha = (B_\alpha^2 + 4(2 - \alpha))^{1/2} - B_\alpha, \quad (1)$$

where

$$B_\alpha = Y_\alpha(1 - Y_\alpha^2)^{\alpha+1} 2(\alpha + 1)(2\alpha + 1)^{\alpha+3/2} (2\alpha)^{-(1+\alpha)}$$

and

$$Y_\alpha = \left(\{(4\alpha^2 + 6\alpha + 1)/(2\alpha + 1)\}^{1/2} - 1 \right) / (2\alpha + 2).$$

If $\|f\|_\alpha < K_\alpha$ then f is univalent in D .

The proof of our theorem depends on the following lemma:

LEMMA W (KOROLLAR 1 in [5]). *If*

$$E_\alpha = \{g: g \text{ is holomorphic in } D \text{ and } |g(z)| < K_\alpha(1 - |z|^2)^{-\alpha} \text{ in } D\},$$

then

$$B_\alpha = \min \{ \beta \in \mathbb{R}: |g'(z)| < \beta K_\alpha / (1 - |z|^2)^{1+\alpha} \text{ in } D \text{ for all } g \in E_\alpha \}, \quad (2)$$

where B_α and K_α are defined as in the Theorem.

PROOF OF THE THEOREM. We suppose

$$(1 - |z|^2)^\alpha |f''/f'| < K_\alpha$$

and $\alpha \in (0, 1]$. Hence

$$(1 - |z|^2)^{1+\alpha} |f''/f'|^2 < K_\alpha^2.$$

Putting $g = f''/f'$ we apply Lemma W and obtain

Received by the editors August 10, 1979 and, in revised form, December 4, 1979.
1980 *Mathematics Subject Classification*. Primary 30C55.

$$(1 - |z|^2)^{1+\alpha} |(f''/f')'| < B_\alpha K_\alpha.$$

Therefore

$$(1 - |z|^2)^{1+\alpha} |\{f, z\}| < B_\alpha K_\alpha + \frac{1}{2} K_\alpha^2 = 2(2 - \alpha),$$

where $\{f, z\} = (f''/f')' - \frac{1}{2}(f''/f')^2$ is the Schwarzian derivative. The last inequality follows directly from (1). Beesack [2, (2.6) and (2.8)] has proved: If $\lambda \in [1, 2]$ and

$$(1 - |z|^2)^\lambda |\{f, z\}| < 2(3 - \lambda), \quad (3)$$

then f is univalent. Put $\lambda = 1 + \alpha$ and the Theorem follows.

REMARKS. Our attention is restricted to $K_\alpha > 1$ because Becker [1, Korollar 4.1] proved that f is univalent in D if $\|f\|_1 < 1$. As for $\alpha < 1$, $\|f\|_\alpha > \|f\|_1$ we have that f is univalent in D if $\|f\|_\alpha < 1$.

Yamashita [6, Theorem] has proved that if $\alpha \in [0, 1]$, $\beta = 2^{1+\alpha}$ and if

$$\|f\|_\alpha < k_\alpha = (\beta^2 + 4(2 - \alpha))^{1/2} - \beta \quad (4)$$

then f is univalent in D . In particular $k_0 = 1.464 \dots$ and $k_\alpha > 1$ when $\alpha \in [0, 0.416 \dots)$. By (4) we see that k_α is a strictly decreasing function of β . So comparing (1) with (4) we have $K_\alpha > k_\alpha$ everywhere on $(0, 1]$ since $B_\alpha < 2^{1+\alpha}$ on $(0, 1]$, which can be seen with the help of a result due to Robertson [4, (2.9)], namely $B_\alpha < (1 + \alpha)^{1+\alpha} \alpha^{-\alpha}$ if $\alpha > 0$. In fact $\lim_{\alpha \rightarrow 0+} K_\alpha = 2 > 1.464 \dots = k_0$ and we can prove that $K_\alpha > 1$ when $\alpha \in (0, 0.575 \dots) \supset (0, 0.416 \dots)$. Therefore except at $\alpha = 0$ our result improves the theorem in [6] for both the values of k_α and the range of α . Since the value of B_α (see (2)) in Lemma W is the minimum of all upper bounds β and hence cannot be lowered anymore we expect that our theorem cannot be improved significantly if Beesack's result (3) is still adopted.

At $\alpha = 0$, Kudryashov [3, Theorem 1] proved that f is univalent in D if $\|f\|_0 < M = 3.05 \dots$ where M is a root of the equation,

$$8(x(x-2)^3)^{1/2} - 3(4-x)^2 = 12.$$

The authors wish to thank the referee for his valuable suggestions which improve the presentation of this paper.

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