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PARACOMPACTNESS IN PERFECTLY NORMAL, LOCALLY CONNECTED, LOCALLY COMPACT SPACES

DIANE J. LANE

ABSTRACT. It is shown that, under (MA + $\neg$CH), every perfectly normal, locally compact and locally connected space is paracompact.

In [**Ru**, **Z**] Rudin and Zenor use the continuum hypothesis (CH) to construct a perfectly normal, separable manifold that is not Lindelöf and is therefore not paracompact. *Manifold* here means a locally Euclidean Hausdorff space. Rudin has shown recently [**Ru**] that if Martin's Axiom and the negation of the continuum hypothesis (MA + $\neg$CH) hold, then every perfectly normal manifold is metrizable. In this paper we show that Rudin's technique can be used to obtain a more general result: If (MA + $\neg$CH), then every perfectly normal, locally compact and locally connected space is paracompact. Since locally metrizable paracompact spaces are metrizable, Rudin's theorem follows.

The following theorems will be used.

THEOREM 1 (Z. SZENTMIKLOSSY [**S**]). *If* $(MA + \neg CH)$, *then there is no hereditarily separable, nonhereditarily Lindelöf, compact* (*locally compact*) *Hausdorff space.*

THEOREM 2 (JUHASZ [**J**]). *If* $(MA + \neg CH)$, *then there is no hereditarily Lindelöf, nonhereditarily separable compact* (*locally compact*) *Hausdorff space.*

THEOREM 3 (REED AND ZENOR [**R**, **Z**]). *Every perfectly normal, locally compact and locally connected subparacompact space is paracompact.*

THEOREM 4 (ALSTER AND ZENOR [**A**, **Z**]). *Every perfectly normal, locally compact and locally connected space is collectionwise normal with respect to discrete collections of compact sets.*

The following result was obtained independently by H. Junilla and J. Chaber. A proof can be found in [**C**, **Z**].

THEOREM 5. *A space X is perfect and subparacompact if and only if whenever* $\{W_\beta\}_{\beta<\gamma}$ *is a well-ordered open cover of X, there exists a sequence* $\{\mathcal{U}_n\}_{n\in\omega}$ *of open covers of X with the property that if $x \in X$, there exists $n \in \omega$ such that* $\mathrm{st}(x, \mathcal{U}_n) = \{U \in \mathcal{U}_n | x \in U\}$ *is contained in the first member of* $\{W_\beta\}_{\beta<\gamma}$ *that contains x.*

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We prove

THEOREM 6. *If* $(MA + \neg CH)$, *then every perfectly normal, locally compact and locally connected space is paracompact.*

PROOF. Let X be a perfectly normal, locally compact and locally connected space. Since components of a locally connected space are open, we may assume that X is connected. We prove that X is subparacompact and thus, by Theorem 3, is paracompact.

Since compact subsets of a perfect space are hereditarily Lindelöf, Theorem 2 implies that compact subsets of X are hereditarily separable.

By transfinite induction we choose for each $\alpha < \omega_1$ a subset X_α of X such that

(i) X_α is the countable union of open sets, each of which is connected, hereditarily separable and has compact closure, and

(ii) $\mathrm{cl}(\bigcup_{\beta<\alpha} X_\beta) \subset X_\alpha$.

Since X is locally compact and locally connected we can clearly make such choices unless there is an $\alpha < \omega_1$ such that $\mathrm{cl}(\bigcup_{\beta<\alpha} X_\beta)$ is not Lindelöf.

Assume that $\lambda < \omega_1$ and that X_α has been constructed for $\alpha < \lambda$. Let $C = \mathrm{cl}(\bigcup_{\alpha<\lambda} X_\alpha)$. C is separable. Now suppose that C is not hereditarily separable. Then there exists $S = \{x_\beta\}_{\beta<\omega_1} \subset C$ such that if $\beta < \omega_1$ then $x_\beta \notin \mathrm{cl}\{x_\alpha\}_{\alpha<\beta}$. For each $\beta < \omega_1$ let U_β denote a hereditarily separable open set in X such that $x_\beta \in U_\beta$ and $U_\beta \cap \mathrm{cl}\{x_\alpha\}_{\alpha<\beta} = \varnothing$. Since S is not separable, no countable subcollection of the U_β's covers S. So there is an uncountable subset A of ω_1 such that, if $\beta < \alpha$ in A, then $x_\alpha \notin U_\beta$. Since X is perfect there is a sequence $\{V_n\}_{n\in\omega}$ of open sets such that $\bigcap_{n=1}^{\infty} V_n = (C - \bigcup_{\alpha\in A} U_\alpha)$. For some $n \in \omega$, $A_n = \{\alpha \in A \mid x_\alpha \notin V_n\}$ is uncountable. Observe that $S_n = \{x_\alpha \mid \alpha \in A_n\}$ is a closed discrete subset of X. By Theorem 4 we can separate the points of S_n with a disjoint collection of open sets. But this contradicts the fact that C is separable. We conclude that C is hereditarily separable. Since C is locally compact, Theorem 1 implies that C is hereditarily Lindelöf as well. Therefore we can construct X satisfying (i) and (ii).

Observe that $\bigcup_{\alpha<\omega_1} X_\alpha$ is both open and closed, and since we are assuming X is connected, $\bigcup_{\alpha<\omega_1} X_\alpha = X$. ($\bigcup_{\alpha<\omega_1} X_\alpha$ is open by definition. X is perfect and locally compact and therefore is first countable. Since $\mathrm{cl}(\bigcup_{\beta<\alpha} X_\beta) \subset X_\alpha$ for $\alpha < \omega_1$, there can be no points of X in $\mathrm{cl}(\bigcup_{\alpha<\omega_1} X_\alpha) - \bigcup_{\alpha<\omega_1} X_\alpha$.)

So we have a perfectly normal, locally compact and locally connected space $X = \bigcup_{\alpha<\omega_1} X_\alpha$ where each X_α is open, hereditarily Lindelöf and $\mathrm{cl}(\bigcup_{\beta<\alpha} X_\beta) \subset X_\alpha$. We let $X'_\alpha = X_\alpha - \bigcup_{\beta<\alpha} X_\beta$. In order to show that X is subparacompact we use the characterization of perfect subparacompactness given in Theorem 5.

Let $\{W_\beta\}_{\beta<\gamma}$ be a well-ordered open cover of X. Since each X_α is perfect and subparacompact, for each $\alpha < \omega_1$ there is a sequence $\{\mathcal{U}_{\alpha n}\}_{n\in\omega}$ of open covers of X_α having the property described in Theorem 5 with respect to the open cover $\{W_\beta \cap X_\alpha\}_{\beta<\gamma}$ of X_α. We may assume that $\mathcal{U}_{\alpha(n+1)}$ refines $\mathcal{U}_{\alpha n}$ for $n \in \omega$ and $\alpha < \omega_1$.

For each $\alpha < \omega_1$ we can choose by induction a sequence $\{\mathcal{K}_{\alpha n}\}_{n\in\omega}$ having the following properties.

(a) $\mathcal{K}_{\alpha n}$ is a countable collection of open sets covering X'_α and refining $\mathcal{U}_{\alpha n}$,

(b) if $K \in \mathcal{K}_{\alpha n}$, cl$(K)$ is compact, cl$(K) \subset X_\alpha$ and $K \cap X'_\alpha \neq \varnothing$, and

(c) if $x \in X'_\alpha$ and $\mathcal{F}$ is a finite subcollection of $\bigcup\{\mathcal{K}_{\beta j} | \beta < \alpha$ and $j \in \omega\}$ then there are infinitely many distinct elements of $\mathcal{K}_{\alpha n}$ containing x and not intersecting cl$(\bigcup \mathcal{F})$.

Note that since X is locally connected and normal, there are uncountably many distinct open sets containing a given point and lying within a given neighborhood. This, together with the fact that X_α is Lindelöf, allows us to easily construct $\mathcal{K}_{\alpha n}$ satisfying (c).

Let $\mathcal{K}_\alpha = \bigcup_{n \in \omega} \mathcal{K}_{\alpha n}$; let $\mathcal{K} = \bigcup_{\alpha < \omega_1} \mathcal{K}_\alpha$. Note that $|\mathcal{K}_\alpha| = \omega$ and $|\mathcal{K}| = \omega_1$. For $K \in \mathcal{K}_\alpha$, define $g(K) = \alpha$; observe that $g(K) = \sup\{\beta | K \cap X'_\beta \neq \varnothing\}$.

Define P to be the set of all functions f such that

(1) the domain of f, called $D(f)$, is a finite subset of $\mathcal{K}$,

(2) the range of f, called $R(f)$, is a subset of ω, and

(3) if $H, K \in D(f)$, $g(H) < g(K)$, and $H \cap K \cap X'_{g(H)} \neq \varnothing$, then $f(H) > f(K)$.

Partially order P by defining $f \leqslant g$ provided g extends f.

If $K \in \mathcal{K}_{\alpha n}$, define $F_{Kn} = \{f \in P | D(f) \cap \mathcal{K}_{\alpha n} \cap \{H | f(H) > n\}$ covers $K \cap X'_\alpha\}$. Clearly, F_{Kn} is open in $(P, \leqslant)$. We will prove that F_{Kn} is dense. Suppose $f \in P$. Since $\bar{K} \cap X'_\alpha$ is compact and $\mathcal{K}_{\alpha n}$ has property (c), we can choose a finite collection $\mathcal{G} \subset \mathcal{K}_{\alpha n}$ such that $\mathcal{G}$ covers $K \cap X'_\alpha$, $\mathcal{G} \cap D(f) = \varnothing$, and if $G \in \mathcal{G}$ and $J \in D(f) \cap \{H \in \mathcal{K} | g(H) < \alpha\}$ then $G \cap \bar{J} = \varnothing$. Let

$$m = \max\{n, \max\{f(H) | H \in D(f)\}\}.$$

We choose $h \in P$ such that $D(h) = D(f) \cup \mathcal{G}$, $h \upharpoonright D(f) = f$, and if $G \in \mathcal{G}$ $f(G) = m$. Since h extends f and $h \in F_{Kn}$, F_{Kn} is dense in $(P, \leqslant)$.

The proof that $(P, \leqslant)$ is ccc is identical to the proof given by Rudin.

Since $(P, \leqslant)$ is ccc, $\{F_{Kn}: K \in \mathcal{K}, n \in \omega\}$ has cardinality ω_1, and each F_{Kn} is open and dense in $(P, \leqslant)$, by (MA + $\neg$CH) there is a generic $G \subset P$ which intersects every F_{Kn}. If f and f' belong to G there is an $h \in P$ such that h extends both f and f'. We use this G to find a sequence $\{\mathcal{U}_n\}_{n \in \omega}$ of covers of X satisfying Theorem 5 with respect to the open cover $\{W_\beta\}_{\beta < \gamma}$.

Let $\mathcal{K}' = \{K \in \mathcal{K} | K \in D(f)$ for some $f \in G\}$. Note that if f and f' belong to G and $K \in D(f) \cap D(f')$ then $f(K) = f'(K)$. We define a function $F: \mathcal{K}' \to \omega$ by $F(K) = f(K)$ where $f \in G$ and $K \in D(f)$. Let $D_n = \{K \in \mathcal{K}' | F(K) > n\}$.

For $n \in \omega$, $\alpha < \omega_1$, and $x \in X'_\alpha$, choose $U_{xn} \in D_n \cap \mathcal{K}_{\alpha n}$ with $x \in U_{xn}$. Such a U_{xn} exists since if $x \in X'_\alpha$ there is a $K \in \mathcal{K}_{\alpha n}$ containing x and $G \cap F_{Kn} \neq \varnothing$. Let $\mathcal{U}_n = \{U_{xn} | x \in X\}$.

We claim that $\{\mathcal{U}_n\}_{n \in \omega}$ witnesses the fact that X is (perfectly) subparacompact. To see this, suppose $x \in X$ and $\beta < \omega_1$ is the first ordinal such that $x \in W_\beta$. There is an $\alpha < \omega_1$ such that $x \in X'_\alpha$. By (a) there is an $m \in \omega$ such that st$(x, \mathcal{K}_\alpha m) \subset W_\beta$. Choose $K_x \in \mathcal{K}_\alpha \cap \mathcal{K}'$ such that $x \in K_x$. Let $n = \max\{m, F(K_x)\}$. We show that st$(x, \mathcal{U}_n) \subset W_\beta$.

Suppose that $x \in U_{yn} \in \mathcal{U}_n$. Then $y \in X'_\delta$ for some $\delta < \omega_1$ and $U_{yn} \in D_n \cap \mathcal{K}_{\delta n}$.

Case (i). $\alpha = \delta$. Since $\mathcal{U}_{\alpha n}$ refines $\mathcal{U}_{\alpha m}$, st$(x, \mathcal{K}_{\alpha n}) \subset W_\beta$ and $U_{yn} \subset$ st$(x, \mathcal{K}_{\alpha n})$.

Case (ii). $\alpha > \delta$. Since $U_{yn} \in \mathcal{K}_{\delta n}$ implies that $U_{yn} \subset X'_\delta$, $\alpha > \delta$ contradicts the fact that $x \in U_{yn}$.

Case (iii). $\alpha < \delta$. Since $U_{yn} \in D_n$ we know that $F(U_{yn}) > n > F(K_x)$. However since K_x and U_{yn} are both in $\mathcal{K}'$ and G is generic in $(P, <)$, there is an $f \in P$ such that H_x and U_{yn} are in $D(f)$. Since $\alpha = g(K_x)$ and $\delta = g(U_{yn})$ and $X'_\alpha \cap K_x \cap U_{yn} \neq \varnothing$, $\alpha < \delta$ gives a contradiction to (3).

Thus the sequence $\{\mathcal{U}_n\}_{n \in \omega}$ has the desired property. By Theorem 5, X is subparacompact.

COROLLARY TO THEOREM 6. *If* $(MA + \neg CH)$, *then every component of a perfectly normal, locally compact, locally connected space is Lindelöf.*

This follows from the fact that every locally compact, paracompact space is the free union of σ-compact spaces.

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DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36830