PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, December 1980

ON NOWHERE DENSE CCC P-SETS

ALAN DOW AND JAN VAN MILL

ABSTRACT. We prove that no compact Hausdorff space can be covered by nowhere dense ccc P-sets. As an application it follows that if X is a compact Hausdorff space with a nonisolated P-point then $X \times K$ is not homogeneous for any compact ccc space K.

1. Introduction. All spaces under discussion are Tychonoff.

A subset B of a space X is called a *P-set* whenever the intersection of countably many neighborhoods of B is again a neighborhood of B. It is known that no compact space of π-weight ω_1 can be covered by nowhere dense P-sets **[KvMM]**. In addition, there is a compact space of weight ω_2 which can be covered by nowhere dense P-sets **[KvMM]**. In this note we will show that no compact space can be covered by nowhere dense ccc P-sets. As a consequence it follows that if X is a compact space with a nonisolated P-point then $X \times K$ is not homogeneous for any compact ccc space K.

2. Independent matrices. Let X be a space. An indexed family $\{A_j^i: i \in I, j \in J\}$ is called an *I by J independent matrix* for X provided that

(a) each A_j^i is an open F_σ;

(b) if $i \in I$ and j_0, j_1 are distinct elements of J then $A_{j_0}^i \cap A_{j_1}^i = \varnothing$;

(c) if $F \subset I$ is finite and $\varphi: F \to J$ then $\bigcap_{i \in F} A_{\varphi(i)}^i \neq \varnothing$.

This concept, in a slightly different form, is due to Kunen.

In **[vM$_1$]** it was shown that each compact space in which each nonempty G_δ has nonempty interior contains an ω_1 by ω_1 independent matrix. We need a generalization of this result. As usual, a space is called ccc if each pairwise disjoint collection of nonempty open sets is countable. A space is nowhere ccc if no point has a ccc neighborhood.

2.1. THEOREM. *Suppose that X is nowhere ccc. Then X contains an ω_1 by ω_1 independent matrix.*

PROOF. For each finite subset $F \subset \omega_1$ (possibly empty) we will define an open F_σ, $C_F \subset X$, such that

(i) $C_{F \cup \{\alpha\}} \subset C_F$ for all $\max F < \alpha < \omega_1$;

(ii) $C_{F \cup \{\alpha\}} \cap C_{F \cup \{\beta\}} = \varnothing$ if $\max F < \alpha < \beta < \omega_1$

(as usual, an ordinal is the set of smaller ordinals; we define $\max \varnothing = -1$).

We will induct on the cardinality of F. Define $C_\varnothing = X$.

Received by the editors November 30, 1979 and, in revised form, January 28, 1980.

AMS (MOS) subject classifications (1970). Primary 54D35.

Key words and phrases. P-set, independent matrix, ccc, homogeneous.

Suppose that we have defined C_F for all $F \subset \omega_1$ of cardinality n. Let $\{C_{F\cup\{\alpha\}}: \max F < \alpha < \omega_1\}$ be a "faithfully indexed" collection of pairwise disjoint nonempty open F_σ's of C_F. This completes the induction.

FACT. $C_F \cap C_G \neq \varnothing \to (F \subset G) \vee (G \subset F)$.

We induct on the cardinality of $|F| + |G|$. If $|F| + |G| = 1$ then there is nothing to prove. Suppose that we have proved the Fact for all finite sets F, $G \subset \omega_1$ satisfying $|F| + |G| \leq n - 1$. Now take finite sets S, $T \subset \omega_1$ so that $|S| + |T| \leq n$. Define $S' = S - \{\max S\}$. By (i) we have that $C_S \subset C_{S'}$ and consequently $C_{S'} \cap C_T \neq \varnothing$. By induction hypothesis, $S' \subset T$ or $T \subset S'$. If $T \subset S'$ then we are done, so we may assume that $S' \subset T$. Define $T' = T - \{\max T\}$. By precisely the same argumentation we may conclude that $T' \subset S$. Then clearly

$$(S \cap T) \cup \{\max S\} = S \quad \text{and} \quad (S \cap T) \cup \{\max T\} = T.$$

If $\max S \in T$ or $\max T \in S$ then there is nothing to prove. So assume that this is not true. Then by (ii) we have that $C_S \cap C_T = \varnothing$, which is a contradiction.

Let f: $\omega_1 \to \omega_1 \times \omega_1$ be onto and one-to-one. Define $U^\alpha_\beta = \bigcup \{C_{F\cup\{f^{-1}(\langle\alpha,\beta\rangle)\}}: \max F < f^{-1}(\langle\alpha, \beta\rangle)$ and $f[F] \cap (\{\alpha\} \times \omega_1) = \varnothing\}$. Notice that $C_{\{f^{-1}(\langle\alpha,\beta\rangle)\}} \subset U^\alpha_\beta$. We claim that $\{U^\alpha_\beta: \alpha, \beta < \omega_1\}$ is an ω_1 by ω_1 independent matrix for X. First observe that each U^α_β is an open F_σ being the union of at most countably many open F_σ's.

Now, let us assume that $U^\alpha_\beta \cap U^\alpha_\gamma \neq \varnothing$ for some $\beta \neq \gamma$. Withot loss of generality assume that $f^{-1}(\langle\alpha, \beta\rangle) < f^{-1}(\langle\alpha, \gamma\rangle)$. There are finite sets F_0, $F_1 \subset \omega_1$ so that

(a) $C_{F_0\cup\{f^{-1}(\langle\alpha,\beta\rangle)\}} \cap C_{F_1\cup\{f^{-1}(\langle\alpha,\gamma\rangle)\}} \neq \varnothing$;

(b) $\max F_0 < f^{-1}(\langle\alpha, \beta\rangle)$ and $f[F_0] \cap (\{\alpha\} \times \omega_1) = \varnothing$;

(c) $\max F_1 < f^{-1}(\langle\alpha, \gamma\rangle)$ and $f[F_1] \cap (\{\alpha\} \times \omega_1) = \varnothing$.

Since $f^{-1}(\langle\alpha, \gamma\rangle) \notin F_0 \cup \{f^{-1}(\langle\alpha, \beta\rangle)\}$, by the Fact, $F_0 \cup \{f^{-1}(\langle\alpha, \beta\rangle)\} \subset F_1 \cup \{f^{-1}(\langle\alpha, \gamma\rangle)\}$. Therefore $f^{-1}(\langle\alpha, \beta\rangle) \in F_1$, since $f^{-1}(\langle\alpha, \beta\rangle) \neq f^{-1}(\langle\alpha, \gamma\rangle)$. However, this contradicts (c).

Take $\alpha_1, \ldots, \alpha_n < \omega_1$ so that $\alpha_i \neq \alpha_j$ for $i \neq j$. In addition, take $\beta_i < \omega_1$ $(i \leq n)$ arbitrarily. Put $\gamma_i = f^{-1}(\langle\alpha_i, \beta_i\rangle)$ and without loss of generality assume that $\gamma_1 < \gamma_2 < \cdots < \gamma_n$. Then $C_{\{\gamma_1,\ldots,\gamma_n\}} \subset U^{\alpha_1}_{\beta_1} \cap \cdots \cap U^{\alpha_n}_{\beta_n}$, and since $C_{\{\gamma_1,\ldots,\gamma_n\}} \neq \varnothing$ we find that $U^{\alpha_1}_{\beta_1} \cap \cdots \cap U^{\alpha_n}_{\beta_n} \neq \varnothing$. □

3. The first application. A point $x \in X$ is called a *weak P-point* provided that $x \notin \bar{F}$ for each countable $F \subset X - \{x\}$. Kunen **[K]** proved that there is a weak P-point in ω^* ($= \beta\omega\setminus\omega$). Subsequently van Mill **[vM$_1$]** proved that there is a weak P-point in each compact F-space of weight 2^ω in which each nonempty G_δ has nonempty interior (an *F-space* is a space in which each cozero set is C^*-embedded). Bell **[B]** has since shown that the weight condition is superfluous. Using Theorem 2.1 by precisely the same technique as in **[vM$_1$]** we obtain the following generalization.

3.1. THEOREM. *Each compact nowhere ccc F-space contains a weak P-point.*

4. The main result. In this section we derive our main result. The techniques of proof used in the following lemma is the same as in **[vM$_1$]**, **[vM$_2$]**.

4.1. LEMMA. *No compact nowhere ccc space can be covered by ccc P-sets.*

PROOF. Let X be a compact nowhere ccc space. Clearly X is not finite, so there is a collection $\{V_n: n < \omega\}$ of (faithfully indexed) pairwise disjoint nonempty open F_σ subsets of X. For each $n < \omega$ let $\{U_\alpha^i(n): \alpha < \omega_1, i < \omega\}$ be an ω_1 by ω independent matrix for V_n (Theorem 1.1). Notice that each $U_\alpha^i(n)$ is an open F_σ of X. Put $\mathfrak{F} = \{A \subset X: \forall n < \omega\ \forall i \leq n\ \exists \alpha < \omega_1$ such that $U_\alpha^i(n) \subset A\}$. It is clear that $\mathfrak{F}$ has the finite intersection property, so there is an $x \in \bigcap_{F \in \mathfrak{F}} \bar{F}$. We claim that $x \notin K$ for each ccc P-set K. Indeed, let $K \subset X$ be any ccc P-set. Since K is ccc for each $n < \omega$ and for each $i \leq n$ there is an $\alpha(n, i) < \omega_1$ so that

$$U_{\alpha(n,i)}^i(n) \cap K = \varnothing.$$

Put $F = \bigcup_{n<\omega} \bigcup_{i\leq n} U_{\alpha(n,i)}^i(n)$. Then $F \in \mathfrak{F}$ and F is an open F_σ being the union of countably many open F_σ's. Also, $F \cap K = \varnothing$. Since K is a P-set, it also follows that $\bar{F} \cap K = \varnothing$. We conclude that $x \notin K$. □

We now come to our main result.

4.2. THEOREM. *No compact space can be covered by ccc nowhere dense P-sets.*

PROOF. Let X be a compact space and suppose that X can be covered by ccc nowhere dense P-sets. Let $U \subset X$ be nonempty and open and suppose that U is ccc. Let B be a nowhere dense P-set meeting U. Since $B \cap U$ is nowhere dense in U the fact that U is ccc implies that there is a countable family $\mathcal{G}$ of compact subsets of $U - B$ so that $\bigcup \mathcal{G}$ is dense in U. However, this is impossible since B is a P-set. So U is not ccc. But now the assumption that X can be covered by ccc nowhere dense P-sets contradicts Lemma 4.1. □

5. Another application. A space X is called *homogeneous* provided that for all x, $y \in X$ there is an autohomeomorphism φ from X onto X mapping x onto y. It is well known that although X is not homogeneous the product $X \times K$ can be homogeneous for certain K (for example, let X be a convergent sequence and let K be the Cantor set). This makes the following straightforward corollary to Theorem 4.2 of some interest.

5.1. COROLLARY. *Let X be a compact space having a nonisolated P-point. Then $X \times K$ is not homogeneous for any compact ccc nonempty space K.*

PROOF. Let x be a nonisolated P-point of X. Then $\{x\} \times K$ is a ccc nowhere dense P-set of $X \times K$. Take any $\langle x, y\rangle \in \{x\} \times K$. By Theorem 4.2 there is a point $\langle p, q\rangle \in X \times K$ so that $\langle p, q\rangle \notin E$ for any nowhere dense ccc P-set $E \subset X \times K$. It is clear that no autohomeomorphism of $X \times K$ can map $\langle x, y\rangle$ onto $\langle p, q\rangle$. □

6. Questions. Since there is a compact space X of weight ω_2 which can be covered by nowhere dense P-sets (which all have to have cellularity at most ω_2), Theorem 4.2 suggests the following question:

6.1. QUESTION. *Is there a compact space X which can be covered by nowhere dense P-sets of cellularity at most ω_1?*

Since Frankiewicz and Mills [FM] have shown that Con(ZFC + ω^* can be covered by nowhere dense P-sets) the question naturally arises whether it is consistent that ω^* can be covered by nowhere dense P-sets of cellularity at most ω_1. Let us answer this question.

6.2. PROPOSITION. *ω^* cannot be covered by nowhere dense P-sets of cellularity at most ω_1.*

PROOF. Under CH the result follows from [KvMM]. So assume $\neg$CH. Kunen [K] proved that (in ZFC) there is a 2^ω by 2^ω independent matrix of clopen subsets of ω^*. Since $\omega_1 < 2^\omega$ we can use the same proof as in Lemma 4.1 to get a point $x \in \omega^*$ so that $x \notin B$ for any P-set B of cellularity at most ω_1. $\square$

Let us finally notice that Proposition 5.1 suggests the following question.

6.3. QUESTION. *Let X be a compact space having a nonisolated P-point and let K be compact. Is $X \times K$ not homogeneous?*

REFERENCES

[B] M. Bell, *A compactification $w_{\mathfrak{B}}(N)$ of N with $w_{\mathfrak{B}}(N)\setminus N$ ccc nonseparable* (to appear).

[FM] R. Frankiewicz and C. F. Mills, *More on nowhere dense closed P-sets* (to appear).

[K] K. Kunen, *Weak P-points in N^**, Proc. Bolyai János Soc. Colloq. Topology, (Budapest, 1978) (to appear).

[KvMM] K. Kunen, J. van Mill and C. F. Mills, *On nowhere dense closed P-sets*, Proc. Amer. Math. Soc. **78** (1980), 119–123.

[vM$_1$] J. van Mill, *Weak P-points in compact F-spaces* (to appear).

[vM$_2$] ______, *A remark on the Rudin-Keisler order of ultrafilters*, Houston J. Math. (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MANITOBA R3T 2N2, CANADA (Current address of Alan Dow)

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803

Current address (Jan van Mill): Department of Mathematics, Vrije Universiteit, Amsterdam, The Netherlands