INVARIANT SETS FOR CLASSES OF MATRICES OF ZEROS AND ONES

RICHARD A. BRUALDI¹ AND JEFFREY A. ROSS

ABSTRACT. Let $\mathfrak{A}(R, S)$ denote the class of all $m \times n$ matrices of 0's and 1's with row sum vector R and column sum vector S . A set $I \times J$ ($I \subseteq \{1, \dots, m\}$, $J \subseteq \{1, \dots, n\}$) is said to be invariant if each matrix in $\mathfrak{A}(R, S)$ contains the same number of 1's in the positions $I \times J$. We prove that if there are no invariant singletons, then an invariant set $I \times J$ satisfies $I = \emptyset$, $I = \{1, \dots, m\}$, $J = \emptyset$ or $J = \{1, \dots, n\}$.

Let m and n be positive integers and let $R = (r_1, \dots, r_m)$ and $S = (s_1, \dots, s_n)$ be nonnegative integral vectors. Let $\mathfrak{A}(R, S)$ denote the class of all $m \times n$ matrices of 0's and 1's with row sum vector R and column sum vector S . Conditions for $\mathfrak{A}(R, S)$ to be nonempty have been given by Gale [2] and Ryser [3], [4, p. 63]. In addition, the nonempty class $\mathfrak{A}(R, S)$ has been extensively investigated (see Chapter 6 of [4] and the references therein, see also [1]).

Let A be an $m \times n$ matrix of 0's and 1's. An *interchange* is a transformation which replaces the 2×2 submatrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1}$$

with the 2×2 submatrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

or vice-versa. If the submatrix (1) lies in rows p, q and columns u, v of A , then we call the interchange a $(p, q; u, v)$ -interchange. An interchange does not alter the row or column sum vectors of a matrix. Given $A, B \in \mathfrak{A}(R, S)$, Ryser [4, p. 68] has proved that there is a sequence of interchanges which transforms A into B .

Let A be an $m \times n$ matrix and let $I \subseteq \{1, \dots, m\}$, $J \subseteq \{1, \dots, n\}$. Then $A[I, J]$ denotes the submatrix of A whose rows are indexed by I and whose columns are indexed by J . If $\bar{I} = \{1, \dots, m\} - I$ and $\bar{J} = \{1, \dots, n\} - J$, then $A(I, J) = A[\bar{I}, \bar{J}]$. For a matrix X of 0's and 1's, $\sigma(X)$ denotes the number of 1's of X .

Consider a nonempty class $\mathfrak{A}(R, S)$ and let $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$. Then $I \times J$ is an *invariant set* for $\mathfrak{A}(R, S)$ provided

$$\sigma(A[I, J]) = \sigma(B[I, J])$$

Received by the editors November 1, 1979.

AMS (MOS) subject classifications (1970). Primary 05B20, 05C50.

¹Research partially supported by N.S.F. Grant No. MCS 76-06374 A01.

© 1980 American Mathematical Society
0002-9939/80/0000-0639/\$02.25

for all $A, B \in \mathfrak{A}(R, S)$. Since

$$\sigma(A(I, J)) = \sum_{i \in \bar{I}} r_i - \sum_{j \in J} s_j + \sigma(A[I, J]),$$

it follows that if $I \times J$ is an invariant set for $\mathfrak{A}(R, S)$ then so is $\bar{I} \times \bar{J}$. Similarly, $I \times \bar{J}$ and $\bar{I} \times J$ are invariant sets whenever $I \times J$ is. Clearly $I \times J$ is an invariant set for $\mathfrak{A}(R, S)$ whenever one of the following holds: $I = \emptyset$, $I = \{1, \dots, m\}$, $J = \emptyset$, $J = \{1, \dots, n\}$. We refer to such invariant sets as *trivial*; all other invariant sets are *nontrivial*. An *invariant position* is an invariant set of cardinality one. Thus the position (i, j) is invariant provided all or none of the matrices in $\mathfrak{A}(R, S)$ have their (i, j) -entry equal to 1. Ryser's invariant 1 [4, p. 69] corresponds to an invariant position (i, j) for which the (i, j) -entry of each matrix in $\mathfrak{A}(R, S)$ equals 1.

THEOREM. Let $R = (r_1, \dots, r_m)$ and $S = (s_1, \dots, s_n)$ be nonnegative integral vectors such that $\mathfrak{A}(R, S)$ is nonempty. If $\mathfrak{A}(R, S)$ has a nontrivial invariant set, then it has an invariant position.

PROOF. Suppose $\mathfrak{A}(R, S)$ has a nontrivial invariant set. Thus there exists an invariant set $I \times J$ such that $1 < |I| < m - 1$, $1 < |J| < n - 1$ and $|I| + |J|$ is maximum. If $|I| = m - 1$ and $|J| = n - 1$, then (i, j) is an invariant position where $\bar{I} = \{i\}$ and $\bar{J} = \{j\}$. Thus we may suppose that $|J| < n - 2$. Let $k \in \bar{J}$ and let $J' = J \cup \{k\}$. Then $|J'| < n - 1$, and it follows that $I \times J'$ is not an invariant set. Thus there exists a matrix $A = [a_{ij}] \in \mathfrak{A}(R, S)$ and a single interchange which transforms A to B such that

$$\sigma(A[I, J']) > \sigma(B[I, J']). \quad (2)$$

Since $I \times J$ is an invariant set, this interchange is a $(p, q; k, l)$ -interchange where $p \in I$, $q \in \bar{I}$, and $l \in \bar{J}'$. By (2), $a_{pk} = 1$. We partition J into two sets J_1, K_1 such that $a_{pj} = 1$ for $j \in J_1$ and $a_{pj} = 0$ for $j \in K_1$. We partition I into four sets I_1, I_2, I_3, I_4 and $\bar{I}$ into four sets N_1, N_2, N_3, N_4 such that

$$\begin{aligned} a_{ik} &= 1, & a_{il} &= 0 & \text{for } i \in I_1 \text{ or } i \in N_1, \\ a_{ik} &= 0, & a_{il} &= 1 & \text{for } i \in I_2 \text{ or } i \in N_2, \\ a_{ik} &= 1, & a_{il} &= 1 & \text{for } i \in I_3 \text{ or } i \in N_3, \\ a_{ik} &= 0, & a_{il} &= 0 & \text{for } i \in I_4 \text{ or } i \in N_4. \end{aligned}$$

Then $p \in I_1$ and $q \in N_2$.

Let $i \in N_2$ and suppose that $a_{ij} = 0$ for some $j \in J_1$. Then the $(p, i; j, l)$ -interchange reduces the number of 1's in the set $I \times J$ of positions. Since $I \times J$ is an invariant set, it follows that $A[N_2, J_1] = 1$, a matrix of all 1's. In a similar way it follows that $A[N_2, K_1] = 0$ and then that $A[I_1, J_1] = 1$ and $A[I_1, K_1] = 0$. Now let $i \in I_2$ and suppose that $a_{ij} = 0$ for some $j \in J_1$. Then the $(p, i; j, l)$ -interchange followed by the $(p, q; j, k)$ -interchange increases the number of 1's in the set of positions $I \times J$. Hence $A[I_2, J_1] = 1$ and, by a similar argument, $A[I_2, K_1] = 0$. Now suppose that $i \in I_3$ and $a_{ij} = 0$ for some $j \in J_1$. Then the $(i, q; j, k)$ -interchange increases the number of 1's in the set $I \times J$ of positions. It follows that

$A[I_3, J_1] = 1$. Similarly, $A[I_4, K_1] = 0$. Since $\bar{I} \times J$ is also an invariant set, it follows that $A[N_1, J_1] = 1$, $A[N_1, K_1] = 0$, $A[N_3, J_1] = 1$ and $A[N_4, K_1] = 0$. Hence A has the form given in (3).

$$\begin{array}{c}
 J_1 \quad K_1 \quad k \quad l \\
 \begin{array}{c} I_1 \\ I_2 \\ I_3 \\ I_4 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{array} \left[\begin{array}{c|c|c|c|c}
 1 & 0 & 1 & 0 & \\
 1 & 0 & 0 & 1 & \\
 1 & & 1 & 1 & \\
 & 0 & 0 & 0 & \\
 1 & 0 & 1 & 0 & \\
 1 & 0 & 0 & 1 & \\
 1 & & 1 & 1 & \\
 & 0 & 0 & 0 &
 \end{array} \right]
 \end{array} \quad (3)$$

Since $J \neq \emptyset$, either $J_1 \neq \emptyset$ or $K_1 \neq \emptyset$. We first suppose that $K_1 \neq \emptyset$. Define a partition P_1, Q_1 of $\{1, \dots, m\}$ by

$$P_1 = I_1 \cup I_2 \cup I_4 \cup N_1 \cup N_2 \cup N_4, \quad Q_1 = I_3 \cup N_3.$$

Let $t \geq 1$ and suppose sets $P_t \subseteq \{1, \dots, m\}$, $Q_t \subseteq \{1, \dots, m\}$ and $J_t \subseteq \{1, \dots, n\}$ have been defined. We then show how to define sets P_{t+1} , Q_{t+1} and J_{t+1} . Let J'_t be the largest subset of $\{1, \dots, n\} - (J_t \cup K_1)$ such that $A[Q_t, J'_t] = 1$. Let J''_t be the subset of $\{1, \dots, n\}$ consisting of all those $j \notin J_t \cup J'_t \cup K_1$ such that $A[P_t, \{j\}]$ contains a 1. Then $J_{t+1} = J_t \cup J'_t \cup J''_t$. Let P'_t be the largest subset of Q_t such that the matrix $A[P'_t, J''_t]$ contains a 0 in every row. For $t = 0$ we define $P'_0 = P_1$. Then $P_{t+1} = P_t \cup P'_t$ and $Q_{t+1} = Q_t - P'_t$. It follows from the definitions that

$$\emptyset \neq P_1 \subseteq P_2 \subseteq \dots, \quad (4)$$

$$J_1 \subseteq J_2 \subseteq \dots, \quad (5)$$

$$Q_1 \supseteq Q_2 \supseteq \dots, \quad (6)$$

$$P_t, Q_t \text{ is a partition of } \{1, \dots, m\}, \quad t = 1, 2, \dots, \quad (7)$$

$$K_1, J_t \text{ are disjoint subsets of } \{1, \dots, n\}, \quad t = 1, 2, \dots, \quad (8)$$

$$A[P_t, \{1, \dots, n\} - (J_{t+1} \cup K_1)] = 0. \quad (9)$$

We have already shown that $A[Q_1, J_1] = 1$. An inductive argument using the facts that $A[Q_t, J'_t] = 1$ (by choice of J'_t) and $A[Q_t - P'_t, J''_t] = 1$ (by choice of P'_t) proves that

$$A[Q_t, J_t] = 1, \quad t = 1, 2, \dots \quad (10)$$

We now verify the following for $t = 1, 2, \dots$ by induction on t :

For each matrix $B = [b_{rs}] \in \mathfrak{A}(R, S)$ such that $B[P_1, K_1] = 0$ and

$$B[\{1, \dots, m\}, J_t] = A[\{1, \dots, m\}, J_t],$$

we have $B[P_t, K_1] = 0$.

(11)

For $t = 1$, (11) is trivial. Let $t \geq 1$ and suppose (11) holds for t . Let $B[P_t, K_1] = \mathbf{0}$ and $B[\{1, \dots, m\}, J_{t+1}] = A[\{1, \dots, m\}, J_{t+1}]$. Since $J_t \subseteq J_{t+1}$, it follows from the inductive assumption that $B[P_t, K_1] = \mathbf{0}$. Suppose $B[P_{t+1}, K_1] \neq \mathbf{0}$. Then it follows that $B[P'_t, K_1] \neq \mathbf{0}$. Thus there exist $i \in P'_t$ and $j \in K_1$ such that $b_{ij} = 1$. By definition there exists a $v \in J'_t$ such that $a_{iv} = 0$. Since $J'_t \subseteq J_{t+1}$, it follows that $b_{iv} = 0$. It follows from (9) and the definition of J'_t that there exists a $u \in P'_{t-1}$ such that $a_{uv} = 1$ and hence $b_{uv} = 1$. Since $B[P_t, K_1] = \mathbf{0}$ and $P'_{t-1} \subseteq P_t$, $b_{uj} = 0$. The $(i, u; j, v)$ -interchange applied to B yields a matrix $C = [c_{rs}]$ satisfying

$$C[\{1, \dots, m\}, J_t] = A[\{1, \dots, m\}, J_t].$$

Suppose first that $t \geq 2$. Then $P_1 \cap (P'_t \cup P'_{t-1}) = \emptyset$ and hence $C[P_1, K_1] = \mathbf{0}$. It follows from the inductive assumption that $C[P_t, K_1] = \mathbf{0}$. Since $c_{uj} = 1$ where $u \in P_t$ and $j \in K_1$, we have a contradiction. Now let $t = 1$. Since $k, l \in J'_1$ and $j \in K_1$ and $v \in J'_1$, it follows that $C[\{1, \dots, m\}, J_1 \cup K_1 \cup \{k, l\}]$ is the same as $A[\{1, \dots, m\}, J_1 \cup K_1 \cup \{k, l\}]$ except that $C[P_1, K_1]$ and $A[P_1, K_1]$ differ in exactly one position as do $C[Q_1, K_1]$ and $A[Q_1, K_1]$. Thus either

$$A[\{p\}, J_1 \cup K_1 \cup \{k, l\}] = C[\{p\}, J_1 \cup K_1 \cup \{k, l\}]$$

or

$$A[\{q\}, J_1 \cup K_1 \cup \{k, l\}] = C[\{q\}, J_1 \cup K_1 \cup \{k, l\}].$$

We are now in a position to apply our previous argument to conclude that C has the form (3). Thus

$$\mathbf{0} = C[P_1, K_1] \neq A[P_1, K_1] = \mathbf{0},$$

a contradiction. This proves (11). In particular,

$$A[P_t, K_1] = \mathbf{0}, \quad t = 1, 2, \dots \quad (12)$$

Let s be the smallest positive integer such that $J_s = J_{s+1}$. Then it follows that $P_s = P_{s+1}$ and $Q_s = Q_{s+1}$. Suppose $A[P_s, \{1, \dots, n\} - (K_1 \cup J_s)] \neq \mathbf{0}$. Then there exists $e \in \{1, \dots, n\} - (K_1 \cup J_s)$ such that $A[P_s, \{e\}] \neq \mathbf{0}$. If $A[Q_s, \{e\}] = 1$, then $e \in J'_s$. If $A[Q_s, \{e\}] \neq 1$, then $e \in J''_s$. Since $J_{s+1} = J_s \cup J'_s \cup J''_s$, we have a contradiction and it follows that

$$A[P_s, \{1, \dots, n\} - (K_1 \cup J_s)] = \mathbf{0}. \quad (13)$$

It follows from (12) and (13) that

$$A[P_s, \{1, \dots, n\} - J_s] = \mathbf{0},$$

and from (10), that $A[Q_s, J_s] = 1$. Since by (7), P_s, Q_s is a partition of $\{1, \dots, m\}$, A has the form

$$\begin{array}{c} J_s \quad \bar{J}_s \\ P_s \left[\begin{array}{c|c} & \mathbf{0} \\ \hline Q_s & 1 \end{array} \right]. \end{array} \quad (14)$$

Since $p \in P_1 \subseteq P_s$, $P_s \neq \emptyset$. Since we have assumed that $K_1 \neq \emptyset$, we conclude that $\bar{J}_s \neq \emptyset$. It now follows from (14) that each position of the nonempty set $P_s \times \bar{J}_s$ is an invariant position of $\mathfrak{A}(R, S)$.

A similar argument in the case that $J_1 \neq \emptyset$ completes the proof of the theorem.

We note that for $m, n > 1$, the converse of the theorem follows trivially. Necessary and sufficient conditions for the class $\mathfrak{A}(R, S)$ to have an invariant position follow from Theorem 3.2 of [4, p. 69].

REFERENCES

1. R. A. Brualdi and J. A. Ross, *On Ryser's maximum term rank formula*, Linear Algebra and Appl. **29** (1980), 33–38.
2. D. Gale, *A theorem on flows in networks*, Pacific J. Math. **7** (1957), 1073–1082.
3. H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Canad. J. Math. **9** (1957), 371–377.
4. ———, *Combinatorial mathematics*, The Carus Mathematical Monographs, No. 14, The Mathematical Association of America; distributed by Wiley, New York, 1963.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706 (Current address of R. A. Brualdi)

Current address (J. A. Ross): Department of Mathematics, Computer Science and Statistics, University of South Carolina, Columbia, South Carolina 29208