A NOTE ON SUM-FREE SETS OF INTEGERS

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ABSTRACT. A set S of integers is said to be sum-free if $a, b \in S$ implies $a + b \notin S$. Let $g(n, k)$ denote the cardinality of a largest subset of $\{1, 2, \dots, n\}$ that can be partitioned into k sum-free sets. In this note we show that $g(n, 2) = n - [n/5]$.

Let k be an integer. A set S of integers is said to be sum-free modulo k if $a, b \in S$ implies $a + b \equiv x \pmod{k}$ has no solution in S . A set which is sum-free modulo 0 is customarily called sum-free. Let $g(n, k)$ denote the cardinality of a largest subset of $\{1, 2, \dots, n\}$ that can be partitioned into k sum-free subsets, and let $h(k)$ denote the largest integer m for which there exists some way of partitioning $\{1, 2, \dots, m\}$ into k subsets which are sum-free modulo $m + 1$.

The numbers $g(n, k)$ and $h(k)$ were introduced in [1], where H. L. Abbott and E. T. H. Wang proved, among other things, that

- (1) $g(n, k) \geq n - [n/(h(k) + 1)]$,
- (2) $g(n, 1) = n - [n/(h(1) + 1)] = [(n + 1)/2]$,
- (3) $g(n, 2) = n - [n/(h(2) + 1)] = n - [n/5]$ for $n \leq 54$.

They also conjectured that the equality sign holds in (1). In this note we shall prove this conjecture for the case $k = 2$.

THEOREM. $g(n, 2) = n - [n/5]$.

PROOF. A proof of $g(n, 2) \geq n - [n/5]$ was given in [1]. We include it here for completeness. For an arbitrary fixed positive integer n , let

$$N = \{1, 2, \dots, n\}, \quad A = \{p \in N: p \equiv 1 \text{ or } 4 \pmod{5}\}, \\ B = \{p \in N: p \equiv 2 \text{ or } 3 \pmod{5}\}.$$

If $a, b \in A$, then $a + b \equiv 0, 2 \text{ or } 3 \pmod{5}$. This implies $a + b \notin A$. Thus A is sum-free. Similarly, B is sum-free. since $A \cup B$ has $n - [n/5]$ elements, $g(n, 2) \geq n - [n/5]$ by definition.

Now we show that $g(n, 2) \leq n - [n/5]$. Assume that $S \subset N$ can be partitioned into two sum-free subsets S_1 and S_2 . Let $R = N \setminus S$. Write $n = 5q + j$, $0 \leq j < 5$. Our aim is to show that R contains at least q elements. We may assume that S has no less than $4q + j + 1$ elements. Then either S_1 or S_2 has at least $2q + 1$ elements. Without loss of generality, we assume that $S_1 \supset \{x_0 < x_1 < \dots < x_{2q}\}$. Let $D = \{x_r - x_0: r = 1, \dots, 2q\}$. Clearly $D \subset N$ and D has $2q$ elements. Since

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S_1 is sum-free $D \cap S_1 = \emptyset$. So $D \subset S_2 \cup R$. If $D \cap R$ has no less than q elements we are done. So we assume that $D \cap R$ contains less than q elements, then $D \cap S_2 \supset \{y_0 < y_1 < \cdots < y_q\}$, where, for each s with $0 < s < q$, $y_s = x_{r_s} - x_0$ for some r_s with $1 < r_s < 2q$. Let $z_t = y_t - y_0$, $t = 1, \dots, q$. Then $z_t \in N$ and $z_{t_1} < z_{t_2}$ if $t_1 < t_2$. Since S_2 is sum-free, $z_t \notin S_2$. Also

$$z_t = (x_{r_t} - x_0) - (x_{r_0} - x_0) = x_{r_t} - x_{r_0} \notin S_1.$$

So $z_t \in R$ for all $t = 1, \dots, q$, and hence R has at least q elements. This completes the proof.

REFERENCES

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