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SOME PROPERTIES OF CLOSED 1-FORMS ON A SPECIAL RIEMANNIAN MANIFOLD

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ABSTRACT. Let M be a compact Riemannian manifold whose sectional curvature is strictly negative; then every closed 1-form on M has a singularity.

1. Introduction. Let M be a compact Riemannian manifold. If we assume that the dimension of the manifold is even and the sectional curvature of M is negative δ-pinched, where $\delta > (n^2 + 5n + 7)/(n^2 + 6n + 14)$, then every harmonic 1-form on M has a singularity [9], where $n = \dim M$.

The main purpose of the present paper is to generalize the above result. Now this result can be stated as follows. Let M be a compact Riemannian manifold whose sectional curvature is strictly negative; then every closed 1-form on M has a singularity.

2. We consider a compact Riemannian manifold M, whose sectional curvature is strictly negative. We assume that there is a closed 1-form w on M without singularities. Then from [10, p. 153] we obtain that the manifold is the total space of a fibre bundle whose base manifold is a circle, that is

$$F \xrightarrow{i} M \xrightarrow{p} S. \tag{2.1}$$

From this fibre bundle we have the following exact sequence [5, p. 153]:

$$\cdots \xrightarrow{p_*} \pi_{n+1}(S, b_0) \xrightarrow{d_*} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(M, e_0) \xrightarrow{p_*} \pi_n(S, b_0)$$
$$\to \cdots \xrightarrow{p_*} \pi_1(S, b_0) \xrightarrow{d_*} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(M, e),$$

where $b_0 \in S$ is a basic point and the fibre $F = p^{-1}(b_0)$ is not empty, which is called a basic fibre and choose $e_0 \in F$ as a basic point.

From the above exact sequence we obtain

$$\pi_2(S, b_0) \xrightarrow{d_*} \pi_1(F, e) \xrightarrow{i_*} \pi_1(M, e_0) \xrightarrow{p_*} \pi_1(S, b_0) \xrightarrow{d_*} \pi_0(F, e_0). \tag{2.2}$$

Since $\pi_1(S, b_0) = \mathbf{Z}$, $\pi_2(S, b_0) = 0$ and we can assume $\pi_0(F, e) = 0$, [3], then the sequence (2.2) takes the form:

$$0 \xrightarrow{d_*} \pi_1(F, e) \xrightarrow{i_*} \pi_1(M, e_0) \xrightarrow{p_*} \mathbf{Z} \to 0. \tag{2.3}$$

From the exact sequence (2.3) we conclude that the mapping i_* is injective and the mapping p_* is surjective.

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The exact sequence (2.3) can be written

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3, \tag{2.4}$$

where $G_1 = \pi_1(F, e)$, $G_2 = \pi_1(M, e_0)$, $G_3 = \mathbf{Z}$, $i_* = f_1$ and $p_* = f_2$.

We assume that F is not simply connected. That means $G_1 = \pi_1(F, e)$ is not trivial.

From the exact sequence (2.4) and since f_1 is injective and f_2 surjective we can consider that G_1 is a normal subgroup of G_2 and the quotient group G_2/G_1 is isomorphic to G_3.

Since G_3 is an infinite cycle we obtain that G_2 considered as a set has the form

$$(\ldots, a^{-2}G_1, a^{-1}G_1, G_1, aG_1, a^2G_1, \ldots),$$

or the form

$$(\ldots, G_1a^{-2}, G_1a^{-1}, G_1, G_1a, G_1a^2, \ldots),$$

where a is an element of the group G_2.

Since G_1 is a normal subgroup of G_2 we have $a^{-1}G_1a = G_1$; that is, there is an element a of G_2 such that $a^{-1}G_1a \cap G_1 = G_1 \supset \{e\}$ where e is the identity element of G_2.

The following proposition is known [**1**, p. 46].

PROPOSITION (2.1). *Let M be a compact Riemannian manifold whose sectional curvature is strictly negative. If $G = \pi_1(M, m, \gamma)$ is a ray subgroup of $\pi_1(M, m)$ and $\beta \notin G$, then $\beta G \beta^{-1} \cap G = \{e\}$, where e is the identity element of $\pi_1(M, m)$.*

From the above and Proposition (2.1) we have a contradiction. We have reached a contradiction because we have assumed that the Riemannian manifold M admits a closed 1-form without singularities. If F is simply connected, then $\pi_1(F, e)$ is trivial and $G_2 = \pi_1(M, e_0) = \mathbf{Z}$, which is not true for a compact manifold with negative sectional curvature [**7**].

Therefore we obtain the theorem.

THEOREM (2.1). *Let M be a compact Riemannian manifold whose sectional curvature is strictly negative. Then every closed 1-form on M has at least one singularity.*

From this theorem we have the corollary.

COROLLARY (2.1). *Let M be a compact Riemannian manifold whose sectional curvature is strictly negative. Then every harmonic 1-form on M has at least one singularity.*

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