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FURTHER DIVISIBILITY PROPERTIES OF THE q-TANGENT NUMBERS

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ABSTRACT. The q-tangent number $T_{2n+1}(q)$ is shown to be divisible by $(1+q)^{a(n,1)}(1+q^2)^{a(n,2)}\cdots(1+q^n)^{a(n,n)}$, where the $a(n,i)$'s are positive integers having the maximal property that $a(n,1)+a(n,2)+\cdots+a(n,n)=2n$ whenever n is a power of 2.

1. Introduction. The q-tangent numbers are polynomials that may be defined by

$$\sum_{n\geq 0} T_{2n+1}(q)x^{2n+1}/(q;q)_{2n+1}$$
$$=\Big(\sum_{n\geq 0}(-1)^n x^{2n+1}/(q;q)_{2n+1}\Big)\Big/\Big(\sum_{n\geq 0}(-1)^n x^{2n}/(q;q)_{2n}\Big), \tag{1.1}$$

where $(a;q)_n=(1-aq)\dots(1-aq^{n-1})$ for $n\geq 1$ and $(a;q)_0=1$. When q equals 1, the q-tangent numbers become the ordinary tangent numbers T_{2n+1} ($n\geq 0$) occurring in the Taylor expansion of $\tan x$

$$\sum_{n\geq 0} T_{2n+1}x^{2n+1}/(2n+1)!=\tan x. \tag{1.2}$$

Because of the relation

$$(n+1)T_{2n+1}=2^{2n}G_{2n+2}, \tag{1.3}$$

where G_{2n+2} is an *odd* integer called the Genocchi number (see e.g. [3]), Schützenberger [6] raised the problem of finding a polynomial of the form

$$\prod_{i\geq 1}(1+q^i)^{a(n,i)}$$

that divides $T_{2n+1}(q)$. Along these lines Andrews and Gessel [2] proved that $T_{2n+1}(q)$ is divisible by

$$AG_n(q)=(1+q)^{[n/2]+1}(1+q^2)\dots(1+q^n). \tag{1.4}$$

The purpose of this paper is to extend the result of the latter work as follows.

Every integer n may be written as $n=m2^l$ with m odd and $l\geq 0$, so that the polynomial

$$Ev_n(q)=\prod_{0\leq j\leq l}(1+q^{m2^j}) \tag{1.5}$$

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may be defined (and is always divisible by $(1 + q^n)$). Let

$$D_n(q) = \prod_{1 \leqslant i \leqslant n} Ev_i(q) \quad \text{for } n \text{ odd},$$
$$= (1 + q^2) \prod_{1 \leqslant i \leqslant n} Ev_i(q) \quad \text{for } n \text{ even}, \tag{1.6}$$

or, in an equivalent manner,

$$D_1(q) = 1 + q, \qquad D_2(q) = (1 + q)^2(1 + q^2)^2 \tag{1.7}$$

and for $n \geqslant 3$

$$D_n(q) = D_{n-2}(q) Ev_{n-1}(q) Ev_n(q). \tag{1.8}$$

For convenience, let $Ev_0(q) = D_0(q) = 1$.

THEOREM 1. *For each $n \geqslant 0$ the polynomial $D_n(q)$ is a divisor of $T_{2n+1}(q)$.*

As $(1 + q)(1 + q^{n-1})(1 + q^n)$ divides $Ev_{n-1}(q)Ev_n(q)$ for every $n \geqslant 2$, formula (1.8) shows by induction that $AG_n(q)$ (given in (1.4)) divides $D_n(q)$ for all n. Thus Theorem 1 extends the result obtained by Andrews and Gessel. Now to compare the divisibility properties of $AG_n(q)$ with $D_n(q)$ write the latter polynomial in the form

$$D_n(q) = \prod_{1 \leqslant i \leqslant n} (1 + q^i)^{a(n,i)}. \tag{1.9}$$

The first values of the coefficients $a(n, i)$ $(1 \leqslant i \leqslant n)$ are shown in Table 1.

n \ i	1	2	3	4	5	6	7	8
1	1							
2	2	2						
3	2	1	1					
4	3	3	1	1				
5	3	2	1	1	1			
6	3	3	2	1	1	1		
7	3	2	2	1	1	1	1	
8	4	4	2	2	1	1	1	1

TABLE 1

The number of factors in $AG_n(q)$ (resp. $D_n(q)$) is $[n/2] + n$ (resp. $a(n, 1) + \cdots + a(n, n)$).

PROPOSITION 2. *If $2^l \leqslant n < 2^{l+1}$ $(l \geqslant 1)$, then*

$$(a(n, 1) + \cdots + a(n, n)) - ([n/2] + n) \geqslant 2^{l-1} - 1. \tag{1.10}$$

Finally, as $D_n(1)$ divides the tangent number T_{2n+1}, it follows from (1.9) and (1.3), since G_{2n+2} is odd, that

$$a(n, 1) + \cdots + a(n, n) \leqslant 2n. \tag{1.11}$$

Whenever equality holds, we may say that $D_n(q)$ is *maximal.*

PROPOSITION 3. *When n is a power of 2, then*

$$a(n, 1) + \cdots + a(n, n) = 2n, \tag{1.12}$$

i.e. $D_n(q)$ is maximal.

The pattern of the paper by Andrews and Gessel is followed closely. In particular, the crucial part is played by a divisibility property of Gaussian polynomials (Lemma 2.2) that, roughly speaking, sorts the cyclotomic polynomials ϕ_d according to the parity of d.

2. A divisibility property of Gaussian polynomials. The polynomials $Ev_n(q)$ defined in (1.5) can be expressed in terms of cyclotomic polynomials ϕ_d as follows.

LEMMA 2.1. *For each $n \geqslant 1$ we have*

$$Ev_n(q) = \prod \{\phi_d(q): d|2n, d \text{ even}\}. \tag{2.1}$$

PROOF. Let $n = m2^l$ with m odd and $l \geqslant 0$. For each $j = 0, 1, \ldots, l$ consider the set

$$A_j = \{d: d|m2^{j+1}, d \nmid m2^j\}$$

and let

$$B = \{d: d|m2^{l+1}, d \text{ even}\}.$$

By definition

$$1 - q^i = \prod_{d|i} \phi_d(q) \quad \text{for each } i \geqslant 1. \tag{2.2}$$

As $(1 - q^{2i}) = (1 - q^i)(1 + q^i)$, we then derive

$$1 + q^i = \prod \{\phi_d(q): d|2i, d \nmid i\}. \tag{2.3}$$

In particular, if $0 \leqslant j \leqslant l$, then

$$1 + q^{m2^j} = \prod \{\phi_d(q): d|m2^{j+1}, d \nmid m2^j\} = \prod_{d \in A_j} \phi_d(q).$$

On the other hand, as the sets A_j are two by two disjoint, it suffices to show that B is the union of the A_j's.

But if $d|m2^{j+1}, d \nmid m2^j$ for some j with $0 \leqslant j \leqslant l$, then $d|2n$ (equal to $m2^{l+1}$) and d is even. Thus d belongs to B. Conversely, suppose $d|2n$ and d even. Then $d = m'2^{j+1}$ with m' odd, $m'|m$ and $0 \leqslant j \leqslant l$. Consequently, d is an element of A_j. Q.E.D.

For each $n \geqslant 1$ let

$$Od_n(q) = \prod \{\phi_d(q): d|2n, d \text{ odd}\}, \tag{2.4}$$

so that

$$1 - q^{2n} = Od_n(q)Ev_n(q). \tag{2.5}$$

On the other hand, let the *Gaussian polynomial* be defined by

$$\begin{bmatrix} N \\ M \end{bmatrix} = (q;\, q)_N / ((q;\, q)_M (q;\, q)_{N-M}) \quad \text{for } 0 < M < N,$$
$$= 0 \quad \text{otherwise.}$$

LEMMA 2.2. *For nonnegative integers k and n the expression*

$$\begin{bmatrix} 2n \\ 2k+1 \end{bmatrix} \frac{Ev_0(q)Ev_1(q)\ldots Ev_k(q)}{Ev_{n-k}(q)Ev_{n-k+1}(q)\ldots Ev_n(q)} \tag{2.6}$$

is a polynomial in q.

PROOF. The expression (2.6) is zero if $k > n$. Assume that $0 < k < n - 1$. Using (2.5) the Gaussian polynomial $[{}_{2k+1}^{2n}]$ may be factorized as a product of two factors,

$$\begin{bmatrix} 2n \\ 2k+1 \end{bmatrix} = \frac{Od_n(q)(1 - q^{2n-1})Od_{n-1}(q)\ldots(1 - q^{2n-2k+1})Od_{n-k}(q)}{(1 - q^{2k+1})Od_k(q)(1 - q^{2k-1})\ldots Od_1(q)(1 - q)}$$
$$\cdot \frac{Ev_n(q)Ev_{n-1}(q)\ldots Ev_{n-k}(q)}{Ev_k(q)Ev_{k-1}(q)\ldots Ev_1(q)}.$$

When numerators and denominators are expressed in terms of cyclotomic polynomials, the first factor, because of (2.2) and (2.4) (resp. the second factor, because of (2.1)) only involves cyclotomic polynomials ϕ_d with *d odd* (resp. *d even*). As $[{}_{2k+1}^{2n}]$ is a polynomial and the cyclotomic polynomials are irreducible, each of these two factors is also a polynomial. But the first one is precisely equal to the expression given in (2.6). Q.E.D.

3. Proof of Theorem 1. Let $T(x)$ be the generating function for the q-tangent numbers as written in (1.1). Andrews and Gessel [**2**, p. 282] found that

$$T(x) = (-i)((-ix;\, q)_\infty - (ix;\, q)_\infty) / ((-ix;\, q)_\infty + (ix;\, q)_\infty) \tag{3.1}$$

where $(a;\, q)_\infty = \lim_n (a;\, q)_n$. As $(a;\, q)_\infty = (1 - a)(aq;\, q)_\infty$, it is straightforward to obtain

$$T(x) - T(qx) = x + xT(qx)T(x); \tag{3.2}$$

that is,

$$\sum_{n>0} T_{2n+1}(q)x^{2n} / (q;\, q)_{2n} = 1 + \left(\sum_{k>0} T_{2k+1}(q)q^{2k+1}x^{2k+1} / (q;\, q)_{2k+1}\right)$$
$$\cdot \left(\sum_{j>0} T_{2j+1}(q)x^{2j+1} / (q;\, q)_{2j+1}\right).$$

Equating coefficients of x^{2n} in both members we find that

$$T_{2n+1}(q) = \sum_{0<k<n-1} \begin{bmatrix} 2n \\ 2k+1 \end{bmatrix} q^{2k+1} T_{2k+1}(q) T_{2n-2k-1}(q) \quad (n > 1). \tag{3.3}$$

The proof of Theorem 1 is now completed as follows. First $T_1(q) = T_{2\times 0+1}(q) = 1$. Proceed by induction on $n > 1$. For $0 < k < n - 1$ the expression

$$\begin{bmatrix} 2n \\ 2k+1 \end{bmatrix} \frac{T_{2k+1}(q)T_{2n-2k-1}(q)}{Ev_1(q)\ldots Ev_n(q)} \tag{3.4}$$

is a polynomial because it may be factorized as

$$\begin{bmatrix} 2n \\ 2k+1 \end{bmatrix} \frac{Ev_0(q)\ldots Ev_k(q)}{Ev_{n-k}(q)\ldots Ev_n(q)} \cdot \frac{T_{2k+1}(q)}{Ev_0(q)\ldots Ev_k(q)} \cdot \frac{T_{2n-2k-1}(q)}{Ev_1(q)\ldots Ev_{n-k-1}(q)},$$

the first factor being a polynomial by Lemma 2.2; the other two are also by the induction hypothesis. This shows, when n is odd, that each term in the sum on the right side of (3.3) is divisible by $D_n(q) = Ev_1(q)Ev_2(q)\ldots Ev_n(q)$. Hence, $D_n(q)$ divides $T_{2n+1}(q)$. When n is even, rewrite (3.3) by grouping the terms two by two to give

$$T_{2n+1}(q) = \sum_{0<k<n/2-1} \begin{bmatrix} 2n \\ 2k+1 \end{bmatrix} q^{2k+1}(1 + q^{2(n-2k-1)}) T_{2k+1}(q) T_{2n-2k-1}(q). \tag{3.5}$$

As n is even, $(1 + q^{2(n-2k-1)})$ is divisible by $(1 + q^2)$, and by (3.4) the expression $[{2n \atop 2k+1}]T_{2k+1}(q)T_{2n-2k-1}(q)$ is divisible by $Ev_1(q)Ev_2(q)\ldots Ev_n(q)$. Hence, each term in the sum on the right side of (3.5) is divisible by $D_n(q) = (1 + q^2)Ev_1(q)\ldots Ev_n(q)$. This completes the proof of Theorem 1.

4. Proofs of Propositions 2 and 3. Note that $D_2(q) = (1 + q^2)Ev_1(q)Ev_2(q) = (1 + q)^2(1 + q^2)^2$. Thus (1.12) holds for $n = 2$. Let $n = 2^l$ $(l > 2)$ and proceed by induction on l. Clearly

$$Ev_{2i}(q) = Ev_i(q)(1 + q^{2i}) \qquad (i > 1).$$

Hence,

$$\begin{aligned} D_n(q) &= (1 + q^2) \prod_{1<i<n} Ev_i(q) \\ &= (1 + q^2) \prod_{\substack{1<i<n \\ i \text{ odd}}} Ev_i(q) \cdot \prod_{\substack{1<i<n \\ i \text{ even}}} Ev_i(q) \\ &= (1 + q^2) \prod_{1<i<n/2} (1 + q^{2i-1}) \cdot \prod_{1<i<n/2} Ev_{2i}(q) \\ &= (1 + q^2) \prod_{1<i<n/2} (1 + q^{2i-1}) \cdot \prod_{1<i<n/2} (1 + q^{2i}) \cdot \prod_{1<i<n/2} Ev_i(q). \end{aligned}$$

By grouping the first and last factors we obtain

$$D_n(q) = D_{n/2}(q) \prod_{1<i<n} (1 + q^i). \tag{4.1}$$

Therefore, if the number of factors in $D_{n/2}(q)$ is n, the polynomial $D_n(q)$ will have $n + n = 2n$ factors. This completes the proof of Proposition 3.

As for Proposition 2 let $d_n = a(n, 1) + \cdots + a(n, n)$ for $n > 1$. From (1.6) and (1.9) it follows that $d_{2n} = d_{2n+1}$ for $n > 1$. On the other hand, the number of factors in $AG_{2n+1}(q)$ is $3n + 1$. Let $p = 2^l < 2n < 2^{l+1}$ $(l > 1)$. To prove Proposition 2 it suffices to show that $d_{2n} - (3n + 1) > 2^{l-1} - 1$, i.e.

$$d_{2n} > 3n + 2^{l-1}. \tag{4.2}$$

From (1.8)

$$D_{2n}(q) = D_p(q) \prod_{p+1 \leqslant i \leqslant 2n} Ev_i(q).$$

But, when i is even, the polynomial $Ev_i(q)$ is a product of at least two binomials $(1 + q^j)$. Hence

$$d_{2n} \geqslant d_p + 3(2n - p)/2 = 3n + 2^{l-1},$$

which is inequality (4.2).

5. Concluding remarks. Recall that a permutation $x_1x_2 \ldots x_{2n+1}$ of the sequence $1\ 2 \ldots (2n + 1)$ is *alternating* if $x_1 > x_2, x_2 < x_3, \ldots, x_{2n} < x_{2n+1}$. As $\left[{2n \atop 2k+1}\right]$ is the generating polynomial for permutations $x_1x_2 \ldots x_{2n}$ of $1^{2k+1}2^{2n-2k-1}$ by number of inversions (see e.g. [**1**, p. 41]), it is clear that the running term on the right side of (3.3) is the generating polynomial for alternating permutations $x_1x_2 \ldots x_{2n+1}$ of $1\ 2 \ldots (2n + 1)$ with $x_{2k+2} = 1$, by number of inversions, a result known to several authors [**4**]–[**7**]. The proof of Theorem 1 shows that the latter generating function is itself divisible by $Ev_1(q)Ev_2(q) \ldots Ev_n(q)$. It would be interesting to have a combinatorial proof of this result by using that alternating permutation set-up.

From (3.1) Andrews and Gessel [**2**] derived the recurrence formula

$$T_{2n+1}(q) + \sum_{1 \leqslant j \leqslant n} (-q; q)_{2j-1} \begin{bmatrix} 2n + 1 \\ 2j \end{bmatrix} (-1)^j T_{2n+1-2j}(q)$$
$$= (-1)^n (-q; q)_{2n}. \tag{5.1}$$

It was not possible to use (5.1) directly to prove Theorem 1 because, for instance, when $n = 2^l$ $(l \geqslant 1)$ the polynomial $(-q; q)_{2n}$ is not divisible by $D_n(q)$. That is why we had to derive the quadratic recurrence formula (3.3).

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