A LIMIT ON THE LOEWY LENGTH OF THE ENDOMORPHISM RING OF A MODULE OF FINITE LENGTH

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ABSTRACT. Let M be a module of finite length over an arbitrary ring A . Let $S_1, S_2, \dots, S_m$ be the nonisomorphic simple composition factors of M and let $[M] = n_1[S_1] + n_2[S_2] + \dots + n_m[S_m]$ denote that S_i occurs n_i times in a composition series for M . As a generalization of Schur's lemma we have the following well-known result: If M is nonzero and all the n_i are equal to one then the Loewy length of the endomorphism ring of M is one. We will generalize this fact showing that in general the Loewy length of the endomorphism ring of M is less than or equal to the maximum of the $n_i, i = 1, \dots, m$.

Throughout this note M will denote a module of finite length over an arbitrary ring A . We will denote by $S_1, S_2, \dots, S_m$ the set of nonisomorphic composition factors of M , and we will let $[M] = n_1[S_1] + \dots + n_m[S_m]$ denote that S_i occurs n_i times as a composition factor in a composition series of M . It is well known that $\text{rad}(\text{End}(M))$, the radical of $\text{End}(M)$, is nilpotent [1, p. 324]. We will let $\text{LL}(\text{End}(M))$ denote the Loewy length of such a ring, i.e. $\text{LL}(\text{End}(M))$ is the integer n such that $\text{rad}^n(\text{End}(M)) = 0$, but $\text{rad}^{n-1}(\text{End}(M)) \neq 0$. Further, we will let $\text{Ker } f$ and $\text{Im } f$ denote the kernel and the image of a map f respectively. A classical result in this situation is Schur's lemma and a direct generalization of it is the following easily verified result: Let M be a module of finite length over a ring A and assume $[M] = [S_1] + \dots + [S_m]$, then $\text{LL}(\text{End}(M)) = 1$.

We are going to generalize this to obtain the following general result.

THEOREM. *Let M be a nonzero module of finite length over a ring A and assume $[M] = n_1[S_1] + \dots + n_m[S_m]$. Then*

$$\text{LL}(\text{End}(M)) \leq \max\{n_i | i = 1, \dots, m\}.$$

To prove this Theorem we will start out with a Lemma in which we need the following notion.

A subset B of a ring A is called nilpotent in case there is an integer $t > 0$ such that $x_1 x_2 \dots x_t = 0$ for all sequences $x_1, x_2, \dots, x_t$ in B .

LEMMA. *Let M be a module of finite length over a ring A and assume $[M] = n_1[S_1] + \dots + n_m[S_m]$ with all $n_i \neq 0$.*

(i) *Let $\{f_i\}_{i \in I}$ be a family of maps in $\text{End}(M)$ which is nilpotent. Then $[\cap_i \text{Ker } f_i] = n'_1[S_1] + \dots + n'_m[S_m]$ with all $n'_i \neq 0, i = 1, \dots, m$.*

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- (ii) $[\cap \{\text{Ker } f | f \in \text{rad}(\text{End}(M))\}] = n_1''[S_1] + \cdots + n_m''[S_m]$ with all $n_i'' \neq 0$, $i = 1, \dots, m$.
- (iii) $[M/\Sigma\{\text{Im } f | f \in \text{rad}(\text{End}(M))\}] = n_1'''[S_1] + \cdots + n_m'''[S_m]$ with all $n_i''' \neq 0$, $i = 1, \dots, m$.

PROOF. (i) Assume $\{f_i\}_{i \in I}$ is a nilpotent family of maps in $\text{End}(M)$ and assume $\cap \{\text{Ker } f_i | i \in I\}$ has a composition series with no composition factor isomorphic to S_i for some $i \in \{1, \dots, m\}$. Since M is artin there exists a finite set $\{f_1, \dots, f_k\}$ in $\{f_i\}_{i \in I}$ such that $\cap_{j=1}^k \text{Ker } f_j = \cap \{\text{Ker } f_i | i \in I\}$. Consider now the map $h_1: M \rightarrow M^k$ with coordinates f_j , $j = 1, \dots, k$. Here M^k denotes a direct sum of k copies of M . Then $\text{Ker } h_1 = \cap_{j=1}^k \text{Ker } f_j$. Further let $h_2: (M^k) \rightarrow (M^k)^k$ be the map with coordinates h_1 . Then $\text{Ker } h_2 = (\text{Ker } h_1)^k = (\cap_{j=1}^k \text{Ker } f_j)^k$. This shows that also $\text{Ker } h_2$ has a composition series which contains no composition factor isomorphic to S_i . Therefore the same holds for $\text{Ker } h_2 h_1$. Now by an easy induction argument we obtain for each p a map $h_p: M^{k^{p-1}} \rightarrow M^{k^p}$ such that all coordinates are among the f_j , $j = 1, \dots, k$, and such that $\text{Ker } h_p h_{p-1} \dots h_1$ has no composition series which contains a composition factor isomorphic to S_i . But $\text{Ker } h_p \dots h_1 = \cap \text{Ker } f_{j_p} f_{j_{p-1}} \dots f_{j_1}$ where the intersection is taken over all ordered p -tuples of the maps $f_1, f_2, \dots, f_k$. By assumption the set $\{f_i\}_{i \in I}$ is nilpotent and therefore there exists a p such that all these compositions are zero. But then $\cap \text{Ker } f_p f_{p-1} \dots f_1 = M$ and hence has to have a composition series which contains a composition factor isomorphic to S_i . This is a contradiction. Therefore $\cap \{\text{Ker } f_i | i \in I\}$ has a composition series which for each $i = 1, \dots, m$ contains a composition factor isomorphic to S_i . This completes the proof of (i).

(ii) This follows from (i) since $\text{rad}(\text{End}(M))$ is nilpotent.

(iii) One obtains this result by dualizing the proof of (i).

We are now in a position where we can prove the Theorem. For each j we claim that $M_j = \cap \{\text{Ker } f | f \in \text{rad}^j(\text{End}(M))\}$ has the property that $[M_j] = n_{1,j}[S_1] + \cdots + n_{m,j}[S_m]$ where $n_{i,j} > \min\{n_i, j\}$. This will be proved by induction using the Lemma. By the Lemma this holds for $j = 1$. Assume now that the claim is true for $j < k$ and consider $M_{k+1} = \cap \{\text{Ker } f | f \in \text{rad}^{k+1}(\text{End}(M))\}$ and let $[M_{k+1}] = n_{1,k+1}[S_1] + \cdots + n_{m,k+1}[S_m]$. Suppose there is an $i \in \{1, \dots, m\}$ such that $n_{i,k+1} < \min\{n_i, k+1\}$. It is easy to see that $n_{i,k+1} > n_{i,k}$ and by the induction hypothesis $n_{i,k} > \min\{n_i, k\}$. Hence $n_{i,k+1} = n_{i,k} = k$ and $n_i > k$. This means that the module M/M_k has a composition series with a composition factor isomorphic to S_i and that M_{k+1}/M_k has a composition series with no composition factor isomorphic to S_i . But

$$M_{k+1} = \cap \{g^{-1}(M_k) | g \in \text{rad}(\text{End}(M))\}$$

and

$$M_{k+1}/M_k = \cap \{\text{Ker } f' | f' \in \text{rad}(\text{End}(M))\}$$

where $f': M/M_k \rightarrow M/M_k$ is the map induced by f . Now the family $\{f'\}$ of maps in $\text{End}(M/M_k)$ induced by the maps in $\text{rad}(\text{End}(M))$ is a nilpotent family of maps. We can now apply the Lemma to get a contradiction. This completes the proof of the claim. It is easy to see that $\text{LL}(\text{End}(M))$ is equal to the least integer t such that

$M_i = M$. Let $q = \max\{n_i | i = 1, \dots, m\}$ and consider $[M_q] = n_{1,q}[S_1] + \dots + n_{m,q}[S_m]$. Then by the claim $n_{i,q} \geq \min\{n_i, q\} = n_i$ for all $i \in \{1, \dots, m\}$. Hence $M_q = M$ which completes the proof of the Theorem.

The result in this Theorem is also the best one can obtain by only looking at the number of different composition factors in a composition series of a module of finite length as the following example shows.

Let $A = \mathbb{Z}$ be the ring of integers and let $a = p_1^{n_1} \dots p_m^{n_m}$, where $p_1, \dots, p_m$ are different primes. Consider $A/(a)$. Then $[A/(a)] = n_1[\mathbb{Z}/p_1\mathbb{Z}] + \dots + n_m[\mathbb{Z}/p_m\mathbb{Z}]$ and $\text{LL}(\text{End}(A/(a))) = \max\{n_i | i = 1, \dots, m\}$.

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