A COUNTEREXAMPLE TO CONJECTURES OF PAPAKYRIAKOPOULOS AND SWARUP¹

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ABSTRACT. In [5] Swarup states a group theoretic conjecture P2 and shows that $P1 \Rightarrow P2 \Rightarrow$ the Poincaré conjecture, where P1 is a conjecture of Papakyriakopoulos [3]. We give a counterexample to conjecture P2.

In [3], Papakyriakopoulos stated a conjecture P1 and showed that P1 implies the Poincaré conjecture. In [5], Swarup showed that $P1 \Rightarrow P2 \Rightarrow$ the Poincaré conjecture, where P2 is the following.

CONJECTURE. Let $Q_p = \{a_1, b_1, \dots, a_p, b_p; \prod_{i=1}^p [a_i, b_i], [a_1, b_1\tau]\}$, where $p > 1$ and $\tau \in [\Phi_p, \Phi_p]$, Φ_p being the free group on $a_1, b_1, \dots, a_p, b_p$. Then Q_p is a nontrivial free product. In this note we show, by means of a counterexample, that conjecture P2 is false. We take $p = 2$, $\tau = [a_1, b_1]$ and suppose that $Q_2 = A * B$, where A and B are nontrivial. Now we observe that, with a slight change of notation,

$$\begin{aligned} Q_2 &= \{a, b, c, d; [a, b][c, d], [a, bab\bar{a}\bar{b}]\} \\ &= \{a, b, c, d, x, y; bab\bar{x}, bxb\bar{y}, ab\bar{a}\bar{b}[c, d], abab\bar{a}\bar{b}ab\bar{a}\bar{b}\} \\ &= \{a, b, c, d, x, y; bab\bar{x}, bxb\bar{y}, a\bar{x}[c, d], ax\bar{y}xy\bar{x}\} \\ &= \{b, c, d, x, y; bx\bar{y}xy\bar{x}\bar{b}\bar{x}, bxb\bar{y}, xy\bar{y}x\bar{x}^2[c, d]\} \\ &= \{b, c, d, x, y, z; \bar{z}x\bar{y}, bx\bar{y}xy\bar{x}\bar{b}\bar{x}, bxb\bar{y}, xy\bar{y}x\bar{x}^2[c, d]\} \\ &= \{b, c, d, x, z; bz\bar{x}\bar{z}\bar{b}\bar{x}, bxb\bar{x}\bar{z}, zx\bar{z}\bar{x}[c, d]\}. \end{aligned}$$

Let $K = \{z, x, c, d; [z, x][c, d]\}$. Then using the Freiheitssatz (see [2] or [1]), we see that the subgroup of K generated by x, z is free of rank two, and so therefore are the subgroups generated by $zx\bar{z}, x$ and $x, \bar{z}x$, respectively. It follows that Q_2 is an *HNN*-extension (see, e.g., [1]) of K by the stable letter b , and hence that the subgroup of Q_2 generated by z, x, c, d is naturally isomorphic to K . Now K is not cyclic, and from the results of [4] K is indecomposable with respect to free products. It then follows from the Kurosh Subgroup Theorem that K is contained in a conjugate of A or B , say B . By conjugating the original A and B , we may assume that K is contained in B .

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Since b and K generate Q_2 , it follows that $b \notin B$. Now the second relator of the last presentation of Q_2 above gives $bx\bar{b} = \bar{z}x$, i.e. conjugation of the element x of B by b gives the element $\bar{z}x$ of B . Consideration of the normal form of b shows immediately that this implies $b \in B$, which is a contradiction.

Thus Q_2 must be indecomposable with respect to free products.

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