ON SUBHARMONICITY OF THE CAPACITY OF THE SPECTRUM

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ABSTRACT. It is shown that if T_λ is an analytic operator valued function (or if f, g belong to a uniform algebra A) then n th diameters and logarithmic capacity of $\sigma(T_\lambda)$ (or of the set $g(f^{-1}(\lambda))$) are subharmonic functions of λ (on a suitable domain).

1. Introduction. Let $B(X)$ denote the algebra of all bounded operators in a complex Banach space X and let

$$\lambda \rightarrow T_\lambda: G \rightarrow B(X) \quad (1)$$

be an analytic operator function in a complex domain G . The behaviour of the multivalued function

$$\lambda \rightarrow \sigma(T_\lambda) \quad (2)$$

has attracted the attention of several authors:

Brown and Douglas in [4] asked if the spectral radius of (1) or the function (2) itself have any maximum properties. The answer was given by E. Vesentini [10], who proved in particular:

THEOREM 1 (VESENTINI, 1968). *If $\lambda \rightarrow T_\lambda: G \rightarrow B(X)$ is an analytic function then the function*

$$\lambda \rightarrow \log r(T_\lambda): G \rightarrow [-\infty, +\infty)$$

is subharmonic.

This theorem has some interesting applications ([1], [8]), but its following corollary is even more useful [1, Théorème 1.2.2].

THEOREM 1' (AUPETIT, 1976). *Under assumptions of Theorem 1 the function $\lambda \rightarrow \log(\text{diam}(\sigma(T_\lambda)))$: $G \rightarrow [-\infty, +\infty)$ is subharmonic.*

Analogous results hold also in uniform algebras, with spectrum replaced by the set

$$g(f^{-1}(\lambda)) = \{g(m): m \in \mathfrak{M}_A, f(m) = \lambda\}$$

where g, f belong to a uniform algebra A .

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THEOREM 2 (WERMER, 1975, AUPETIT, WERMER, 1978). *Let A be a uniform algebra and let X denote its Shilov boundary. Assume that $g, f \in A$ and $G = \sigma(f) \setminus f(X)$ is nonvoid. Then the following two functions*

$$\lambda \rightarrow \log \max |g(f^{-1}(\lambda))|: G \rightarrow [-\infty, +\infty),$$

$$\lambda \rightarrow \log \text{diam} |g(f^{-1}(\lambda))|: G \rightarrow [-\infty, +\infty),$$

are subharmonic.

Since these results proved to be useful as well (cf. [1, pp. 99–106], [11]), B. Aupetit asked [1] if $\text{diam}(\cdot)$ and $\max|\cdot|$ above could be replaced by some other functions of set argument, more specifically, whether the following functions were subharmonic

$$\lambda \rightarrow \log \delta_k(\sigma(T_\lambda)), \quad k = 0, 1, 2, \dots, \quad (3)$$

$$\lambda \rightarrow \log \text{cap}(\sigma(T_\lambda)), \quad (4)$$

$$\lambda \rightarrow \log \delta_k(g(f^{-1}(\lambda))), \quad k = 0, 1, 2, \dots, \quad (5)$$

$$\lambda \rightarrow \log \text{cap}(g(f^{-1}(\lambda))), \quad (6)$$

where “ δ_k ” and “cap” stand for k th diameter and logarithmic capacity of a compact plane set.

Special cases of these conjectures have been already known. Assuming that all T_λ , $\lambda \in G$, commute, B. Aupetit has shown subharmonicity of (3) and (4) (oral communication). D. Kumagai has proved that (5), (6) are subharmonic, if the algebra A satisfies

$$\partial_1(A \overset{\sim}{\otimes} A) = \partial_1(A) \times X \cup X \times \partial_1(A),$$

where $\partial_1(A)$ denotes some generalized Shilov boundary (cf. [3], [6]).

The aim of this note is to prove these conjectures without any restricting assumptions. We shall prove a little bit more.

THEOREM 3. *Let $\lambda \rightarrow T_\lambda: G \rightarrow B(X)$ be analytic and let w be a complex polynomial in k variables. Then the function*

$$\lambda \rightarrow \log \max |w(\sigma(T_\lambda) \times \dots \times \sigma(T_\lambda))|: G \rightarrow [-\infty, +\infty)$$

is subharmonic.

THEOREM 4. *Let A be a uniform algebra and X its Shilov boundary. Assume that $f, g \in A$ and $G = \sigma(f) \setminus f(X)$ is nonvoid. Then for any complex polynomial $w(z_1, \dots, z_k)$ the function*

$$\lambda \rightarrow \log \max |w(g(f^{-1}(\lambda)) \times \dots \times g(f^{-1}(\lambda)))|: G \rightarrow [-\infty, +\infty)$$

is subharmonic.

The main result is a direct consequence of these theorems:

COROLLARY. *Under assumptions of Theorems 1 and 2 functions (3)–(6) are subharmonic.*

Readers more interested in the background of discussed problems are referred to the book [1].

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2. Operator case. We first show how the Corollary follows from Theorems 3 and 4.

PROOF. As it was noted by B. Aupetit, subharmonicity of (4), (6) follows from that of (3), (5), for $\delta_k(Z) \searrow \text{cap}(Z)$ for any plane compact Z and the limit of nonincreasing sequence of subharmonic functions is also subharmonic (for notions related to capacity or subharmonicity the reader is referred to [9] or to Appendix II in [1]). Let us recall that

$$\log \delta_k(Z) = \left(k + \frac{1}{2} \right)^{-1} \log \max |W_k(Z \times \cdots \times Z)| \quad (7)$$

where

$$W_k(z_1, \dots, z_k) = \prod_{1 \leq i < j \leq k} (z_i - z_j).$$

By Theorems 3, 4 the functions

$$\lambda \rightarrow \log \max |W_k(\sigma(T_\lambda) \times \cdots \times \sigma(T_\lambda))|$$

and

$$\lambda \rightarrow \log \max |W_k(g(f^{-1}(\lambda)) \times \cdots \times g(f^{-1}(\lambda)))|$$

are subharmonic. They are still so after dividing by the positive scalar $\binom{k+1}{2}$; applying (7) we get that (3), (5) are subharmonic. Q.E.D.

For the proof of Theorem 3 we need the following result of Dash and Schecter (it is actually only a special case of their Corollary 1 in [5]).

THEOREM 5 (DASH, SCHECTER, 1970). *Let X be a complex Banach space and let $Y = X \hat{\otimes} \cdots \hat{\otimes} X$ denote complete k -fold projective tensor product. Let $T \in B(X)$ and $T_i = I \otimes \cdots \otimes T \otimes \cdots \otimes I$, $i = 1, \dots, k$. Then operators $T_1, \dots, T_k$ commute and for any complex polynomial $w(z_1, \dots, z_k)$ we have*

$$\sigma(w(T_1, \dots, T_k)) = w(\sigma(T) \times \cdots \times \sigma(T)).$$

PROOF OF THEOREM 3. Denote by $T_i(\lambda)$ the bounded operator in Y , determined uniquely by $I \otimes \cdots \otimes T_\lambda \otimes \cdots \otimes I$, $i = 1, \dots, k$, $\lambda \in G$. Put

$$W_\lambda = w(T_1(\lambda), \dots, T_k(\lambda)).$$

Clearly W_λ is an analytic function of λ . By the Vesentini theorem $\lambda \rightarrow \log r(W_\lambda)$ is subharmonic. On the other hand

$$\begin{aligned} r(W_\lambda) &= \max |\sigma(W_\lambda)| = \max |\sigma(w(T_1(\lambda), \dots, T_k(\lambda)))| \\ &= \max |w(\sigma(T_\lambda) \times \cdots \times \sigma(T_\lambda))| \end{aligned}$$

(by Theorem 5). Thus

$$\lambda \rightarrow \log \max |w(\sigma(T_\lambda) \times \cdots \times \sigma(T_\lambda))|$$

is subharmonic. Q.E.D.

3. Uniform algebras. The following folklore lemma is proved here for the convenience of the reader.

LEMMA. *A and X as in Theorem 4. Assume that $f \in A$ and $\lambda_0 \notin f(X)$. Then there is $\varepsilon > 0$ such that for every $m_0 \in A^*$ with $m_0 \perp (f - \lambda_0)A$ there is an analytic vector function*

$$\lambda \rightarrow m(\lambda): D(\lambda_0, \varepsilon) \rightarrow A^*$$

such that

$$m(\lambda) \perp (f - \lambda)A, \quad m(\lambda_0) = m_0, \quad (8)$$

and

$$\|m(\lambda)\| \leq 2\|m_0\| \quad (9)$$

for $\lambda \in D(\lambda_0, \varepsilon)$.

PROOF. Without loss of generality we can assume that $\lambda_0 = 0$. By [7, Corollary 3.3.7] there is $\Delta > 0$ such that

$$\|fg\| \geq \|g\|/\Delta \quad \text{for all } g \in A. \quad (10)$$

In particular, fA is closed. Consider action of A on A^* defined by

$$(f \cdot m)(g) = m(fg), \quad f, g \in A, m \in A^*.$$

With (10) we have

$$\text{for any } m \in A^* \text{ there is } m' \in A^* \text{ such that } m = f \cdot m' \text{ and } \|m'\| \leq \Delta \|m\|. \quad (*)$$

Indeed. Consider linear functional

$$fg \rightarrow m(g): fA \rightarrow C.$$

Its norm is not greater than $\|m\|\Delta$:

$$|m(g)| \leq \|m\| \cdot \|g\| \leq (\|m\|\Delta) \cdot \|fg\|,$$

so it can be extended to a functional m' on A with the same norm (Hahn-Banach theorem). Such m' satisfies (*). Applying (*) inductively (starting from given m_0) we find $m_n \in A^*$, $n = 1, 2, \dots$, such that

$$m_{n-1} = f \cdot m_n, \quad \|m_n\| \leq \Delta^n \|m_0\|, \quad n = 1, 2, \dots \quad (11)$$

Put $\varepsilon = (2\Delta)^{-1}$. In $D(0, \varepsilon)$ the series

$$m(\lambda) = \sum_{n=0}^{\infty} m_n \lambda^n$$

is absolutely convergent to an analytic vector-function; moreover $\|m(\lambda)\| \leq 2\|m_0\|$.

Using (11) we compute that

$$(f - \lambda)m(\lambda) = fm_0 + \sum_{n=1}^{\infty} (fm_n - m_{n-1})\lambda^n = fm_0.$$

By assumptions m_0 annihilates fA , i.e. $fm_0 = 0$. Therefore

$$(f - \lambda)m(\lambda) \equiv 0, \quad \lambda \in \Delta(\lambda_0, \varepsilon). \quad \text{Q.E.D.}$$

We shall prove Theorem 4 using the following criterion for subharmonicity:

PROPOSITION. *Let $G \subset C$ and $\phi: G \rightarrow [-\infty, +\infty)$ be upper semicontinuous. Assume that for every $\lambda_0 \in G$ there exist $\varepsilon > 0$ and function ψ defined in $D(\lambda_0, \varepsilon)$ such that*

$$(i) \psi(\lambda_0) = \phi(\lambda_0),$$

$$(ii) \psi(\lambda) \leq \phi(\lambda), \lambda \in D(\lambda_0, \varepsilon),$$

(iii) ψ is uniformly bounded from the above Baire class and satisfies the mean-value inequality, i.e.

$$\psi(\lambda_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi(\lambda_0 + re^{i\theta}) d\theta, \quad 0 < r < \varepsilon.$$

Then ϕ is subharmonic in G .

This proposition is an immediate consequence of a classical test for subharmonicity, see [13, §9, 4, p. 58].

PROOF OF THEOREM 4. We have to show that the function

$$\phi(\lambda) = \log \max\{|w(z_1, \dots, z_k)| : z_i \in g(f^{-1}(\lambda)), i = 1, \dots, k\}, \quad (12)$$

is subharmonic in $G = \sigma(f) \setminus f(X)$. Observe first that ϕ is upper semicontinuous, for $\lambda \rightarrow g(f^{-1}(\lambda))$ is an upper semicontinuous set-valued function (we omit the details).

By the proposition it remains to show that for an arbitrary $\lambda_0 \in G$ we can find a positive ε and a function ψ defined in $D(\lambda_0, \varepsilon)$ such that conditions (i)–(iii) are satisfied.

Fix $\lambda_0 \in G$. To define ψ observe first that the maximum in (12) is actually attained, therefore we can choose multiplicative-linear functions $m_1, \dots, m_k \in f^{-1}(\lambda_0)$ such that

$$\phi(\lambda_0) = \log|w(g(m_1), \dots, g(m_k))|.$$

Since $m_i(f) = \lambda_0$, clearly m_i annihilate $(f - \lambda_0)A$ and we can apply the lemma to “initial values” $m_0 = m_i$, $i = 1, \dots, k$. We obtain $\varepsilon > 0$ and k analytic vector-functions

$$\lambda \rightarrow m_i(\lambda): D(\lambda_0, \varepsilon) \rightarrow A^*$$

such that $m_i(\lambda_0) = m_i$ and conditions (8), (9) are satisfied.

Let $B = A \hat{\otimes} A \hat{\otimes} \dots \hat{\otimes} A$ be the k -fold tensor product completed with respect to projective norm. With natural multiplication B becomes a commutative Banach algebra with unit. Put $g_i = e \otimes \dots \otimes g \otimes \dots \otimes e$. We can consider $\tilde{g} := w(g_1, \dots, g_k)$ and $\tilde{g}^n$. Finally we define

$$\psi(\lambda) := \limsup_n \left(\frac{1}{n} \log |\langle \tilde{m}(\lambda), \tilde{g}^n \rangle| \right), \quad \lambda \in D(\lambda_0, \varepsilon), \quad (13)$$

where $\tilde{m}(\lambda)$ is the unique element of B^* determined by $m_1(\lambda) \otimes \dots \otimes m_k(\lambda)$.

To end the proof we have to check that conditions (i)–(iii) are satisfied.

Condition (i). Since $\tilde{m}(\lambda_0) = m_1 \otimes \cdots \otimes m_k$ is a multiplicative-linear functional on B , therefore

$$\begin{aligned} w(\langle m_1, g \rangle, \dots, \langle m_k, g \rangle)^n &= w(\langle \tilde{m}(\lambda_0), g_1 \rangle, \dots, \langle \tilde{m}(\lambda_0), g_k \rangle)^n \\ &= \langle \tilde{m}(\lambda_0), w(g_1, \dots, g_k) \rangle^n \\ &= \langle \tilde{m}(\lambda_0), w(g_1, \dots, g_k)^n \rangle = \langle \tilde{m}(\lambda_0), \tilde{g}^n \rangle. \end{aligned} \quad (14)$$

Thus

$$\limsup_n \frac{1}{n} \log |\langle \tilde{m}(\lambda_0), \tilde{g}^n \rangle| = \log |w(m_1(g), \dots, m_k(g))|,$$

which proves Condition (i).

Condition (ii). Fix $\lambda \in D(\lambda_0, \epsilon)$ and denote by J_λ the closed ideal in B generated by the following k elements:

$$e \otimes e \otimes \cdots \otimes (f - \lambda e) \otimes \cdots \otimes e, \quad i = 1, \dots, k.$$

Since all $m_i(\lambda)$ satisfy conditions (8), (9) of the lemma, $\tilde{m}(\lambda)$ annihilates J_λ and $\|\tilde{m}(\lambda)\| \leq 2^k \|m_1\| \cdots \|m_k\| = 2^k$. Therefore

$$|\langle \tilde{m}(\lambda), \tilde{g}^n \rangle| \leq \|\tilde{m}(\lambda)\| \cdot \|\tilde{g}^n + J_\lambda\| \leq 2^k \|\tilde{g}^n + J_\lambda\|, \quad (15)$$

and

$$\limsup_n \left(\frac{1}{n} \log |\langle \tilde{m}(\lambda), \tilde{g}^n \rangle| \right) \leq \log \left(\limsup_n \sqrt[n]{\|\tilde{g}^n + J_\lambda\|} \right),$$

where $\|\tilde{g}^n + J_\lambda\|$ denotes the norm of the coset $[\tilde{g}^n]$ in the quotient algebra B/J_λ . By the spectral radius formula the last inequality reads

$$\psi(\lambda) \leq \log r_\lambda([\tilde{g}]) \quad (16)$$

where $r_\lambda(\cdot)$ denotes spectral radius with respect to B/J_λ .

The Gelfand space of B/J_λ can be identified with the annihilator of J_λ in $\mathfrak{N}_B$ (cf. [12, Theorem 11.4]). Therefore there are $m'_1, \dots, m'_k \in \mathfrak{N}_A$ such that $\tilde{m} = m'_1 \otimes \cdots \otimes m'_k$ annihilates J_λ and

$$r_\lambda([\tilde{g}]) = |\langle \tilde{m}, \tilde{g} \rangle|. \quad (17)$$

In particular it holds

$$\begin{aligned} 0 &= \langle \tilde{m}, e \otimes \cdots \otimes (f - \lambda e) \otimes \cdots \otimes e \rangle \\ &= \langle m'_i, f - \lambda e \rangle \cdot \prod_{j \neq i} \langle m'_j, e \rangle = f(m'_i) - \lambda, \end{aligned}$$

hence $m'_i \in f^{-1}(\lambda)$, $i = 1, \dots, k$. Similarly as in (14) we can get

$$\langle \tilde{m}, \tilde{g} \rangle = w(g(m'_1), \dots, g(m'_k)),$$

and by definition of ϕ

$$\phi(\lambda) \geq \log |w(g(m'_1), \dots, g(m'_k))| = \log |\langle \tilde{m}, \tilde{g} \rangle|.$$

This relation together with (17) and (16) proves Condition (ii).

Condition (iii). From (15) it follows in particular that

$$|\langle \tilde{m}(\lambda), \tilde{g}^n \rangle| \leq (2^k \|\tilde{g}\|)^n, \quad \lambda \in D(\lambda_0, \epsilon), n = 1, 2, \dots,$$

therefore functions

$$\lambda \rightarrow \frac{1}{n} \log |\langle \tilde{m}(\lambda), \tilde{g}^n \rangle| : D(\lambda_0, \epsilon) \rightarrow [-\infty, +\infty), \quad n = 1, 2, \dots, \quad (18)$$

are uniformly bounded from above. We omit the simple argument which shows that the functions $\lambda \rightarrow \langle \tilde{m}(\lambda), \tilde{g}^n \rangle$ are analytic and conclude that functions (18) satisfy mean value inequality. This and uniform boundedness imply that $\psi(\lambda)$ satisfies the mean-value inequality. Q.E.D.

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