LENGTH ESTIMATES FOR HOLOMORPHIC FUNCTIONS

SHINJI YAMASHITA

ABSTRACT. Let $f(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k$ ($n \geq 1$) be holomorphic in $U: |z| < 1$. MacGregor [1, Theorem 2] proved that if $l(r, f)$ is the length of the outer boundary of the image $D(r, f)$ of the disk: $|z| < r$ by f , then $2\pi r^n |a_n| \leq l(r, f)$ for $0 < r < 1$. We introduce the notion of the exact outer boundary $C^*(r, f)$ of $D(r, f)$ and prove that $2\pi r^n |a_n| \leq l^*(r, f) \leq l(r, f)$ for $0 < r < 1$, where $l^*(r, f)$ is the length of $C^*(r, f)$. We shall make use of the estimate to obtain a criterion for f to be Bloch in U .

In this note f is always a function nonconstant and holomorphic in the disk $U = \{|z| < 1\}$. Let $D(r, f)$ be the image of the disk $\{|z| < r\}$ by f ($0 < r < 1$), namely, $D(r, f)$ is the projection of the Riemannian image of $\{|z| < r\}$ by f , to the plane: $|z| < \infty$. The boundary $C(r, f)$ of the unbounded component of the complement of $D(r, f)$ is called the outer boundary of $D(r, f)$. If, in the definition of $C(r, f)$, we replace $D(r, f)$ by its closure, then we obtain the boundary $C^*(r, f)$, called the exact outer boundary of $D(r, f)$. Let $l(r, f)$ ($l^*(r, f)$, respectively) be the length of $C(r, f)$ ($C^*(r, f)$, resp.). Apparently, $l^*(r, f) \leq l(r, f)$. T. H. MacGregor [1, Theorem 2] proved that, if

$$f(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k \quad (n \geq 1) \quad (1)$$

is the Taylor expansion in U , then

$$2\pi r^n |a_n| \leq l(r, f) \quad (0 < r < 1). \quad (2)$$

We shall call this Theorem M. The following theorem improves Theorem M.

THEOREM 1. *If (1) holds, then*

$$2\pi r^n |a_n| \leq l^*(r, f) \quad (0 < r < 1). \quad (3)$$

Before the proof we propose an example of f such that $l^*(r, f) < l(r, f)$ for an r . Consider $g(z) = (cz + 2/\sqrt{3})^3$, where $c > 1$ is a constant. Then $l^*(c^{-1}, g) < l(c^{-1}, g)$, in other words, $l(c^{-1}, g) - l^*(c^{-1}, g)$ is the length of the "shoreline" of the "bay" of the "island" $D(c^{-1}, g)$. I am indebted to the referee for this example; my example in the first draft was false.

We remember the isoperimetric theorem.

LEMMA. *Of all rectifiable Jordan curves with a given perimeter, the circle has the interior of the maximum area.*

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Since $C^*(r, f)$ is a Jordan curve consisting of a finite number of analytic arcs, it is rectifiable. Let $a(r, f)$ be the area of $D(r, f)$, and let A be the area of the Jordan domain $\mathfrak{D}$ with the boundary $C^*(r, f)$. Then $a(r, f) \leq A$. It follows from the Lemma that

$$A \leq \pi [l^*(r, f) / (2\pi)]^2. \quad (4)$$

On the other hand [1, Theorem 1],

$$\pi r^{2n} |a_n|^2 \leq a(r, f). \quad (5)$$

The inequality (3) now follows from (4) and (5).

The referee proposed the following alternative proof. Let a univalent holomorphic function h map U onto $\mathfrak{D}$, such that $h(0) = f(0)$. Set $c_1 = h'(0)$. Then Theorem M, together with the limiting process, yields that

$$2\pi |c_1| \leq \lim_{s \rightarrow 1-0} l^*(s, h) = l^*(r, f).$$

Since $g(z) \equiv f(rz)$ is subordinate to h , the same argument as in the proof of Theorem M in [1] is available, so that

$$r^n |a_n| = |g^{(n)}(0)|/n! \leq |c_1|.$$

Therefore (3) is established.

Let

$$\Delta(z, r) = \{w \in U; |w - z|/|1 - \bar{z}w| < r\}$$

be the non-Euclidean hyperbolic disk of the non-Euclidean center $z \in U$ and the non-Euclidean radius $\frac{1}{2} \log[(1+r)/(1-r)]$ ($0 < r < 1$). Let $D(r, f, z)$ be the image of $\Delta(z, r)$ by f , and let $l^*(r, f, z)$ be the length of the exact outer boundary of $D(r, f, z)$, that is, $l^*(r, f, z) = l^*(r, g)$ for the composed function

$$g(w) = f((w + z)/(1 + \bar{z}w)) \quad (6)$$

of a variable $w \in U$.

We consider a localization of Theorem 1 in

THEOREM 2. *Let $n \equiv n(z, f) \geq 1$ be the first number such that $f^{(n)}(z) \neq 0$ at $z \in U$. Then*

$$2\pi r^n (1 - |z|^2)^n |f^{(n)}(z)|/n! \leq l^*(r, f, z) \quad (7)$$

for all r , $0 < r < 1$.

Since f is nonconstant, $n(z, f)$ always exists.

For the proof of (7) we apply (3) to g of (6). Since

$$g^{(n)}(0)/n! = (1 - |z|^2)^n f^{(n)}(z)/n!,$$

we obtain (7).

The case $n = 1$ is of particular interest. If $f'(z) \neq 0$, then

$$(1 - |z|^2) |f'(z)| \leq l^*(r, f, z) / (2\pi r), \quad (8)$$

while if $f'(z) = 0$, the estimate (8) is trivial. Therefore, if there exists r , $0 < r < 1$, such that

$$\sup_{z \in U} l^*(r, f, z) < \infty, \quad (9)$$

then f is Bloch ([2], [3, p. 268]) in the sense that

$$\sup_{z \in U} (1 - |z|^2) |f'(z)| < \infty. \quad (10)$$

Conversely, if f is Bloch, then (9) holds for each r , $0 < r < 1$. In effect, let K be the supremum of (10) and let $\mathcal{L}(r, f, z)$ be the length of the boundary of the Riemannian image of $\Delta(z, r)$ by f . Then

$$\begin{aligned} \mathcal{L}(r, f, z) &= \int_{\partial \Delta(z, r)} |f'(w)| |dw| \\ &\leq K \int_{|w|=r} (1 - |w|^2)^{-1} |dw| = 2\pi Kr(1 - r^2)^{-1}. \end{aligned}$$

Actually, $(1 - |w|^2)^{-1} |dw|$ is invariant for the non-Euclidean transformations. Thus, (9) is a consequence of $l^*(r, f, z) \leq \mathcal{L}(r, f, z)$.

By the similar argument we also have a localization of [1, Theorem 1], which we leave as an exercise.

I wish to express my cordial thanks to the referee for his corrections and criticisms. Without his help, the present work would surely have been aborted.

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, FUKAZAWA, SETAGAYA, TOKYO 158, JAPAN