ARENS REGULARITY AND THE ALGEBRA OF DOUBLE MULTIPLIERS

B. J. TOMIUK

ABSTRACT. We show how Arens regularity of a w.c.c. Banach algebra A with a bounded approximate identity is connected with the algebra of double multipliers on A .

1. Introduction. Let A be a semisimple w.c.c. Banach algebra with a bounded approximate identity. Let E be an element of the second conjugate space A^{**} such that E is a left identity for $(A^{**}, \circ')$ and a right identity for $(A^{**}, \circ)$, where $\circ'$ and $\circ$ denote the two Arens products on A^{**} . Let $M(A)$ be the algebra of double multipliers on A . We show that A is Arens regular if and only if $S^{**}(E) = T^{**}(E)$ for all $(S, T) \in M(A)$.

2. Preliminaries. Let A be a Banach algebra. Then A^* and A^{**} will denote the first and second conjugate spaces of A and π the canonical map of A into A^{**} . For any subset S in A , $l(S)$ and $r(S)$ will denote, respectively, the left and right annihilators of S in A . By an ideal we shall always mean a two-sided ideal unless otherwise specified.

For any $a \in A$, let L_a and R_a be, respectively, the left and right multiplication operators determined by a . An element $a \in A$ is called *left weakly completely continuous* (l.w.c.c.) if L_a is weakly completely continuous. A is called l.w.c.c. if every $a \in A$ is l.w.c.c. Similarly an element $a \in A$ is called *right weakly completely continuous* (r.w.c.c.) if R_a is weakly completely continuous, and A is called r.w.c.c. if every $a \in A$ is r.w.c.c. If A is both l.w.c.c. and r.w.c.c., then A is called w.c.c.

A map $T: A \rightarrow A$ is called a *left* (resp. *right*) *multiplier* if $T(ab) = T(a)b$ (resp. $T(ab) = aT(b)$) for all $a, b \in A$. Let $M_l(A)$ (resp. $M_r(A)$) denote the algebra of all bounded linear left (resp. right) multipliers on A . For $a \in A$, $L_a \in M_l(A)$ and $R_a \in M_r(A)$.

An ordered pair (S, T) of mappings from A into itself is called a *double multiplier* if $xS(y) = T(x)y$, for all $x, y \in A$ [6]. Let $M(A)$ denote the algebra of double multipliers (S, T) on A such that $S \in M_l(A)$ and $T \in M_r(A)$. If A is faithful, i.e., $l(A) = r(A) = (0)$, then every double multiplier on A belongs to $M(A)$ [8, Lemma 3.1, p. 269].

Let A be a Banach algebra. The two Arens products on A^{**} are defined in stages as follows. Let $x, y \in A$, $f \in A$ and $F, G \in A^{**}$.

Received by the editors October 24, 1979 and, in revised form, February 21, 1980.

AMS (MOS) subject classifications (1970). Primary 46H20, 46H10; Secondary 46L20.

Key words and phrases. Weakly completely continuous (w.c.c.) Banach algebra, Arens products, Arens regularity, double multipliers.

Define $f \circ x \in A^*$ by $(f \circ x)(y) = f(xy)$.

Define $F \circ f \in A^*$ by $(F \circ f)(x) = F(f \circ x)$.

Define $F \circ G \in A^{**}$ by $(F \circ G)(f) = F(G \circ f)$.

Define $x \circ' f \in A^*$ by $(x \circ' f)(y) = f(yx)$.

Define $f \circ' F \in A^*$ by $(f \circ' F)(x) = F(x \circ' f)$.

Define $F \circ' G \in A^{**}$ by $(F \circ' G)(f) = G(f \circ' F)$.

Then A^{**} is a Banach algebra under the products $F \circ G$ and $F \circ' G$, and π is an algebra isomorphism of A into $(A^{**}, \circ)$ and $(A^{**}, \circ')$.

A Banach algebra A is called *Arens regular* if the two Arens products agree on A^{**} . If a Banach algebra A contains a bounded right approximate identity then $(A^{**}, \circ)$ has a right identity and if A contains a bounded left approximate identity then $(A^{**}, \circ')$ has a left identity. (See [1, Lemma 3.3, p. 855].) If A has a bounded approximate identity then A^{**} has an element E which is simultaneously a right identity for $(A^{**}, \circ)$ and a left identity for $(A^{**}, \circ')$ [5, Proposition 1.3, p. 93]. It is an immediate consequence of the definition of Arens products and [2, Theorem 2, p. 482] that a Banach algebra A is left (right) w.c.c. if and only if $\pi(A)$ is a right (left) ideal of A^{**} under either Arens product.

All algebras under consideration are over the complex field.

3. Direct sum decompositions of A^{**} .

Notation. Let A be a Banach algebra. Set

$$N_A = \{ G \in A^{**} : \pi(x) \circ G = 0 \text{ for all } x \in A \}$$

and

$$N'_A = \{ G \in A^{**} : G \circ' \pi(x) = 0 \text{ for all } x \in A \}.$$

Let R_1^{**} be the radical of $(A^{**}, \circ)$ and R_2^{**} the radical of $(A^{**}, \circ')$.

We observe that $\pi(x) \circ G = \pi(x) \circ' G$ and $G \circ \pi(x) = G \circ' \pi(x)$ for all $x \in A$ and $G \in A^{**}$ [4]. Although the following lemma can be easily deduced from the results in [8] stated for right multipliers, for the sake of completeness we sketch its proof.

LEMMA 3.1 *Let A be a Banach algebra with a bounded left approximate identity. Let E be a left identity in $(A^{**}, \circ')$. Then the following statements hold.*

(i) *For $S \in M_l(A)$, let $F^S = S^{**}(E)$. Then the mapping $S \rightarrow F^S$ is a continuous algebra isomorphism of $M_l(A)$ into $(A^{**}, \circ')$, and for each $S \in M_l(A)$,*

$$\pi(S(x)) = F^S \circ' \pi(x) \quad (x \in A).$$

(ii) *For each $F \in A^{**}$ such that $F \circ' \pi(A) \in \pi(A)$, there exists $S \in M_l(A)$ and $G \in N'_A$ such that $F = S^{**}(E) + G$.*

PROOF. (i) If in the argument in [7, p. 810] we replace the product $*$ by the product $\circ'$ and T by $S \in M_l(A)$ and make the obvious corresponding changes, we will obtain

$$f \circ' F^S = S^*(f) \quad (f \in A^*, S \in M_l(A)). \quad (1)$$

It is easy to see that the mapping $S \rightarrow F^S$ is linear and one-to-one. It is also

continuous since

$$\|F^S\| = \|S^{**}(E)\| \leq \|S^{**}\| \|E\| = \|S\| \|E\|.$$

Let $S_1, S_2 \in M_r(A)$ and $f \in A^*$. Then using (1), we have

$$\begin{aligned} (F^{S_1} \circ ' F^{S_2})(f) &= F^{S_2}(f \circ ' F^{S_1}) = F^{S_2}(S_1^*(f)) \\ &= S_2^{**}(E)(S_1^*(f)) = S_1^{**}(S_2^{**}(E))(f) \\ &= (S_1 S_2)^{**}(E)(f) = F^{S_1 S_2}(f). \end{aligned}$$

Therefore $S_1 S_2 \rightarrow F^{S_1} \circ ' F^{S_2}$. Let $x \in A$, $f \in A^*$ and $S \in M_r(A)$. Then, using (1) again, we have

$$\begin{aligned} S^{**}(\pi(x))(f) &= \pi(x)(S^*(f)) = S^*(f)(x) \\ &= (f \circ ' F^S)(x) = \pi(x)(f \circ ' F^S) \\ &= (F^S \circ ' \pi(x))(f). \end{aligned}$$

Hence

$$\pi(S(x)) = S^{**}(\pi(x)) = F^S \circ ' \pi(x) \quad (x \in A).$$

(ii) Let $F \in A^{**}$ such that $F \circ ' \pi(A) \subset \pi(A)$. Then since $S \rightarrow S^{**}|_{\pi(A)}$ (the restriction of S^{**} to $\pi(A)$) is an isometric algebra isomorphism of $M_r(A)$ onto $M_r(\pi(A))$, there exists $S \in M_r(A)$ such that

$$S^{**}(\pi(x)) = F \circ ' \pi(x) \quad (x \in A).$$

Let $F^S = S^{**}(E)$. By (i),

$$F^S \circ ' \pi(x) = S^{**}(\pi(x)) = \pi(S(x)) = F \circ ' \pi(x) \quad (x \in A).$$

Therefore $(F - F^S) \circ ' \pi(x) = 0$ ($x \in A$) so that $G = F - F^S \in N'_A$ and $F = F^S + G$.

THEOREM 3.2. *Let A be a semisimple r.w.c.c. Banach algebra with a bounded left approximate identity. Let E be a left identity of $(A^{**}, \circ ')$. Then*

- (i) $A^{**} = N'_A \oplus M'$, where $M' = \{S^{**}(E) : S \in M_r(A)\}$.
- (ii) $N'_A = \{G \in A^{**} : G \circ ' E = 0\}$ and $N'_A = R_2^{**}$.
- (iii) $M' = \{G \circ ' E : G \in A^{**}\}$ and M' is a closed left ideal of $(A^{**}, \circ ')$.

PROOF. (i) Since A is r.w.c.c., $\pi(A)$ is a left ideal of $(A^{**}, \circ ')$ so that $F \circ ' \pi(A) \subset \pi(A)$ for every $F \in A^{**}$. Therefore $A^{**} = N'_A + M'$ by Lemma 3.1. Let $F \in N'_A \cap M'$. Then $F = S^{**}(E)$ for some $S \in M_r(A)$ and $\pi(S(x)) = F \circ ' \pi(x) = 0$ for all $x \in A$. Therefore $S = 0$ and hence $F = 0$. Thus $A^{**} = N'_A \oplus M'$.

(ii) Since $G \circ ' F$ is w^* -continuous in F for fixed G [4, p. 116] and $\pi(A)$ is w^* -dense in A^{**} , $N'_A = \{G \in A^{**} : G \circ ' A^{**} = 0\}$ so that N'_A is an ideal of $(A^{**}, \circ ')$. Clearly $N'_A \subset R_2^{**}$. By [3, Theorem 4.6, p. 130], $R_2^{**} \cap \pi(A) = (0)$ and, by hypothesis, $\pi(A)$ is a left ideal of $(A^{**}, \circ ')$. Hence $R_2^{**} \circ ' \pi(A) \subset R_2^{**} \cap \pi(A) = (0)$. Therefore $R_2^{**} \subset N'_A$. Thus $N'_A = R_2^{**}$. That $N'_A \subset \{G \in A^{**} : G \circ ' E = 0\}$ is clear. On the other hand if $G \circ ' E = 0$ then $G \circ ' \pi(x) = G \circ ' (E \circ ' \pi(x)) = (G \circ ' E) \circ ' \pi(x) = 0$ for all $x \in A$ and hence $G \in N'_A$.

(iii) Let $F = S^{**}(E) \in M'$. Then $F \circ ' E = S^{**}(E) \circ ' E$ and, since S^{**} is a left

multiplier on A^{**} , we get $S^{**}(E) \circ' E = S^{**}(E \circ' E) = S^{**}(E) = F$. Conversely suppose that $F = G \circ' E$, $G \in A^{**}$. Then there are $S \in M_l(A)$ and $Q \in N'_A$ such that $F = S^{**}(E) + Q$. Hence $F \circ' E = S^{**}(E) \circ' E + Q \circ' E = S^{**}(E) \circ' E = S^{**}(E)$. As $F \circ' E = F$, we get $F = S^{**}(E)$. Hence $M' = \{G \circ' E: G \in A^{**}\}$. To see that M' is a left ideal of $(A^{**}, \circ')$, let $F \in A^{**}$ and let $S_1 \in M_l(A)$ be such that $F = S_1^{**}(E) + G$, $G \in N'_A$. Then, for every $S \in M_l(A)$, we have $F \circ' S^{**}(E) = S_1^{**}(E) \circ' S^{**}(E)$. Since S_1^{**} is a left multiplier on $(A^{**}, \circ')$ we obtain

$$\begin{aligned} S_1^{**}(E) \circ' S^{**}(E) &= S_1^{**}(E \circ' S^{**}(E)) = S_1^{**}(S^{**}(E)) \\ &= S_1^{**}S^{**}(E) = (S_1S)^{**}(E) \in M'. \end{aligned}$$

That M' is closed follows from the fact that $S \rightarrow S^{**}(E)$ is a homeomorphism of $M_l(A)$ onto M' by Lemma 3.1 and a well-known theorem of Banach.

For an l.w.c.c. Banach algebra, Theorem 3.2 takes the following form (see also [9, p. 11]).

THEOREM 3.2'. *Let A be a semisimple l.w.c.c. Banach algebra with a bounded right approximate identity. Let E be a right identity in $(A^{**}, \circ)$. Then*

- (i) $A^{**} = N_A \oplus M$, where $M = \{T^{**}(E): T \in M_r(A)\}$.
- (ii) $N_A = \{G \in A^{**}: E \circ G = 0\}$ and $N_A = R_1^{**}$.
- (iii) $M = \{E \circ G: G \in A^{**}\}$ and M is a closed right ideal of $(A^{**}, \circ)$.

COROLLARY 3.3. *Let A be a semisimple w.c.c. Banach algebra with a bounded approximate identity. Then $R_1^{**} = R_2^{**}$.*

PROOF. Let E be an element of A^{**} which is both a left identity for $(A^{**}, \circ')$ and a right identity for $(A^{**}, \circ)$. Let $\{e_\alpha: \alpha \in \Omega\}$ be a bounded approximate identity in A . We may assume that $w^*\text{-}\lim_\alpha \pi(e_\alpha) = E$. Let $F \in R_2^{**}$. Since R_2^{**} is an ideal of $(A^{**}, \circ')$, we have that for all $x \in A$, $\pi(x) \circ' F \in R_2^{**}$ and therefore, by Theorem 3.2,

$$(\pi(x) \circ' F) \circ' E = 0 \quad (x \in A).$$

But, since A is w.c.c., $\pi(x) \circ' F \in \pi(A)$ so that

$$(\pi(x) \circ' F) \circ' E = (\pi(x) \circ F) \circ E = \pi(x) \circ F \quad (x \in A).$$

Hence $\pi(x) \circ F = 0$ for all $x \in A$; in particular $\pi(e_\alpha) \circ F = 0$ for all $\alpha \in \Omega$. Therefore, by the left w^* -continuity of the product $\circ$, we obtain that $E \circ F = 0$. Hence $F \in R_1^{**}$. Similarly we can show that if $F \in R_1^{**}$ then $F \in R_2^{**}$. Therefore $R_1^{**} = R_2^{**}$.

4. The main result.

LEMMA 4.1. *Let A be a semisimple Banach algebra with a bounded approximate identity. Let E be an element of A^{**} which is both a left identity for $(A^{**}, \circ')$ and a right identity for $(A^{**}, \circ)$. Let $(S, T) \in M(A)$. Then*

$$S^{**}(E) = T^{**}(E) + G \quad \text{where } G \in N_A.$$

PROOF. Let $F = S^{**}(E)$. Then, by Lemma 3.1, we have $S^{**}(\pi(x)) = F \circ \pi(x)$ for all $x \in A$. Since

$$T^{**}(\pi(x)) \circ \pi(y) = \pi(x) \circ S^{**}(\pi(y)) = \pi(x) \circ F \circ \pi(y),$$

for all $x, y \in A$, the semisimplicity of A implies that

$$T^{**}(\pi(x)) = \pi(x) \circ F \quad (x \in A).$$

Therefore, by [8, Lemma 5.3, p. 271], $F = T^{**}(E) + G$, where $G \in N_A$.

REMARK. Lemma 4.1 corrects an error in [8, Theorem 5.6, p. 272].

THEOREM 4.2. *Let A be a semisimple w.c.c. Banach algebra with a bounded approximate identity. Let E be an element of A^{**} which is both a left identity for $(A^{**}, \circ')$ and a right identity for $(A^{**}, \circ)$. Then A is Arens regular if and only if $S^{**}(E) = T^{**}(E)$ for all $(S, T) \in M(A)$.*

PROOF. Suppose that A is Arens regular. Then A^{**} is semisimple by [9, Corollary 6.2, p. 12]. Therefore, $R_1^{**} = R_2^{**} = (0)$ so that, by Lemma 4.1 and Theorem 3.2', $S^{**}(E) = T^{**}(E)$ for every $(S, T) \in M(A)$.

Conversely suppose that $S^{**}(E) = T^{**}(E)$ for all $(S, T) \in M(A)$. By Theorems 3.2 and 3.2' (and their notation), we have

$$A^{**} = R_1^{**} \oplus M = R_2^{**} \oplus M'.$$

By Corollary 3.3, $R_1^{**} = R_2^{**}$. We now show that $M = M'$. Let $T \in M_r(A)$. Then, by [8, Lemma 5.1, p. 270],

$$\pi(T(x)) = T^{**}(\pi(x)) = \pi(x) \circ T^{**}(E) \quad (x \in A).$$

Since $\pi(A)$ is a left ideal of $(A^{**}, \circ')$, there exists $S \in M_l(A)$ such that

$$\pi(S(x)) = T^{**}(E) \circ \pi(x) \quad (x \in A).$$

Since $\pi(xS(y)) = \pi(T(x)y)$ for all $x, y \in A$, we have that $(S, T) \in M(A)$. By hypothesis, $S^{**}(E) = T^{**}(E)$. Therefore $M \subset M'$. Similarly we can show that for every $S \in M_l(A)$ there exists $T \in M_r(A)$ such that $S^{**}(E) = T^{**}(E)$ and $(S, T) \in M(A)$. Therefore $M' \subset M$. Hence $M = M'$.

Now let $F_1, F_2 \in M$. Let (S_1, T_1) and (S_2, T_2) be the elements of $M(A)$ defined by F_1 and F_2 , respectively. That is, $\pi(S_i(x)) = F_i \circ \pi(x)$ and $\pi(T_i(x)) = \pi(x) \circ F_i$ for all $x \in A$ and $i = 1, 2$. We have that S_1S_2 is defined by $F_1 \circ' F_2$ and T_2T_1 is defined by $F_1 \circ F_2$. Since $(S_1S_2, T_2T_1) \in M(A)$ [8, p. 270], it follows that

$$\pi(x) \circ (F_1 \circ' F_2) \circ \pi(y) = \pi(x) \circ (F_1 \circ F_2) \circ \pi(y) \quad (x, y \in A).$$

Let $\{e_\alpha: \alpha \in \Omega\}$ be a bounded approximate identity in A . Since $\pi(x) \circ (F_1 \circ' F_2)$ and $\pi(x) \circ (F_1 \circ F_2)$ are in $\pi(A)$, letting $y = e_\alpha$ and taking the limit over e_α , we obtain

$$\pi(x) \circ (F_1 \circ' F_2) = \pi(x) \circ (F_1 \circ F_2) \quad (x \in A).$$

Therefore $F_1 \circ' F_2 - F_1 \circ F_2 \in R_1^{**}$. As $F_1 \circ' F_2 - F_1 \circ F_2 \in M$ and $M \cap R_1^{**} = (0)$, we see that

$$F_1 \circ' F_2 = F_1 \circ F_2$$

so that the two Arens products agree on M . As they clearly agree on R_1^{**} and as $A^{**} = R_1^{**} \oplus M$, it follows that they agree on A^{**} . Therefore A is Arens regular.

COROLLARY 4.3. *Let A be a semisimple w.c.c. Banach algebra with a bounded approximate identity. Then the following statements are equivalent.*

- (i) A is Arens regular.
- (ii) $(A^{**}, \circ)$ is semisimple.
- (iii) $(A^{**}, \circ')$ is semisimple.

PROOF. If A is Arens regular then, by [9, Corollary 6.2, p. 12], A^{**} is semisimple in each of the Arens products. By Corollary 3.3, $R_1^{**} = R_2^{**}$ so that $(A^{**}, \circ)$ is semisimple if and only if $(A^{**}, \circ')$ is semisimple. If $R_1^{**} = R_2^{**} = (0)$ then $S^{**}(E) = T^{**}(E)$ for all $(S, T) \in M(A)$. Therefore, by Theorem 4.2, A is Arens regular.

It follows from the results above that if G is an infinite compact abelian group then the group algebra $L^1(G)$ is not Arens regular. In fact $A = L^1(G)$ is a w.c.c. semisimple Banach algebra with a bounded approximate identity [10]. From [3, Theorem 5.5, p. 136] and Corollary 3.3 we obtain that $R_1^{**} = R_2^{**} \neq (0)$. Hence, by Corollary 4.3, A is not Arens regular.

REFERENCES

1. P. Civin and B. Yood, *The second conjugate space of a Banach algebra as an algebra*, Pacific J. Math. **11** (1961), 847–870.
2. N. Dunford and J. T. Schwartz, *Linear operators. Part I: General theory*, Pure and Appl. Math., vol. 7, Interscience, New York and London, 1958.
3. S. L. Gulick, *Commutativity and ideals in the biduals of topological algebras*, Pacific J. Math. **18** (1966), 121–137.
4. J. Hennesfeld, *A note on the Arens products*, Pacific J. Math. **26** (1968), 115–119.
5. ———, *Finding a maximal subalgebra on which the two Arens products agree*, Pacific J. Math. **59** (1975), 93–98.
6. B. E. Johnson, *An introduction to the theory of centralizers*, Proc. London Math. Soc. (3) **14** (1964), 299–320.
7. L. Maté, *Embedding multiplier operators of a Banach algebra B into its second conjugate space B^{**}* , Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **13** (1965), 809–812.
8. B. J. Tomiuk, *Multipliers on Banach algebras*, Studia Math. **54** (1976), 267–283.
9. ———, *On some properties of Segal algebras and their multipliers*, Manuscripta Math. **27** (1979), 1–18.
10. B. J. Tomiuk and B. Yood, *Topological algebras with dense socle*, J. Functional Analysis **28** (1978), 254–277.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTTAWA, OTTAWA, K1N 6N5 ONTARIO, CANADA