A REMARK ON ANALYTIC SETS WITH σ -COMPACT SECTIONS

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ABSTRACT. We show that there exists an analytic set $A \subset \omega^\omega \times \omega^\omega$ and having σ -compact vertical sections such that A contains no analytic set B with compact vertical sections and having the same projection to the first coordinate as A . This answers a question of J. R. Steel.

The problem considered in this note has its genesis in the following question posed by C. Dellacherie: Suppose $A \subset \omega^\omega \times \omega^\omega$ is analytic (Σ_1^1) and has σ -compact vertical sections. Can A be written as $\bigcup_{n \geq 0} A_n$, where each A_n is analytic and has compact vertical sections? In [5], J. R. Steel showed that Dellacherie's question admits a negative answer. Steel then formulated the following question: Suppose $A \subset \omega^\omega \times \omega^\omega$ is analytic and has σ -compact vertical sections. Does there exist an analytic set B such that $B \subset A$, B has compact vertical sections and $\pi(B) = \pi(A)$, where π denotes projection to the first coordinate? We prove in this note that the answer to Steel's question is also negative. Specifically, we prove

THEOREM. *There exists a Σ_1^1 set $A \subset \omega^\omega \times \omega^\omega$ having countable (and hence σ -compact) vertical sections such that whenever B is Σ_1^1 , $B \subset A$ and B has compact vertical sections, then $\pi(B) \neq \pi(A)$.*

PROOF. We follow the notation and terminology of [2]. Fix a Σ_1^1 set $R \subset \omega^\omega \times \omega^\omega \times \omega^\omega$ which is universal for Σ_1^1 subsets of $\omega^\omega \times \omega^\omega$. By [2, 4D.2], fix a Π_1^1 -recursive partial function $d: \omega \times \omega^\omega \rightarrow \omega^\omega$ which parametrizes points in $\Delta_1^1(\alpha) \cap \omega^\omega$.

Define

$$P(\alpha, s) \leftrightarrow \text{Seq}(s) \ \& \ (\forall \beta)(R(\alpha, \alpha, \beta) \rightarrow (\exists i < 1h(s)) \\ (d((s)_i, \alpha) \downarrow \ \& \ d((s)_i, \alpha) = \beta)).$$

Clearly P is Π_1^1 . By the Easy Uniformization Theorem [2, 4B.4] there is a Π_1^1 set G which uniformizes P . Let $f: \omega^\omega \rightarrow \omega^\omega$ be the (partial) function whose graph is G . Then f is a Π_1^1 -recursive partial function. Next define

$$Q(\alpha, \beta) \leftrightarrow f(\alpha) \downarrow \ \& \ (\exists i)(i < 1h(f(\alpha)) \ \& \ d((f(\alpha))_i, \alpha) \downarrow \ \& \ d((f(\alpha))_i, \alpha) = \beta).$$

Now check that

- (a) Q is Π_1^1 ,
- (b) $Q_\alpha = \{ \beta: Q(\alpha, \beta) \}$ is finite, and
- (c) if $R_{\alpha, \alpha} = \{ \beta: R(\alpha, \alpha, \beta) \}$ is finite, then $R_{\alpha, \alpha} \subset Q_\alpha$.

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The last fact is an easy consequence of the Effective Perfect Set Theorem [2, 4F.1].

Finally, define

$$A(\alpha, \beta) \leftrightarrow (\exists n)((\forall i)(\beta(i) = n) \& \neg Q(\alpha, \beta)).$$

Then A is Σ_1^1 and has countable vertical sections. Moreover, since each Q_α is finite, it follows that $\pi(A) = \omega^\omega$. Suppose now that $B \subset A$, B is Σ_1^1 and B has compact vertical sections. Find a code α_0 for B ; that is, find α_0 such that $B = R_{\alpha_0}$ ($= \{(\alpha, \beta): R(\alpha_0, \alpha, \beta)\}$). Since $B_\alpha \subset A_\alpha$ and A_α is discrete, it follows that B_α is finite. In particular, $B_{\alpha_0} = R_{\alpha_0, \alpha_0}$ is finite. So, by (c), $B_{\alpha_0} \subset Q_{\alpha_0}$. But also $B_{\alpha_0} \subset A_{\alpha_0} \subset \omega^\omega - Q_{\alpha_0}$. It follows that $B_{\alpha_0} = \emptyset$ and hence $\pi(B) \neq \omega^\omega = \pi(A)$. This completes the proof.

We conclude with some remarks.

A. The above shows that the following weak 'reduction' property fails for analytic sets:

If A_n , $n > 0$, are analytic subsets of ω^ω with $\bigcup_{n>0} A_n = \omega^\omega$, then there exist analytic sets B_n , $n > 0$, such that $(\forall n)(B_n \subset A_n)$, $\limsup B_i = \emptyset$ and $\bigcup_{n>0} B_n = \omega^\omega$.

B. With the same notation as in the proof of the Theorem, define $A^* = (\omega^\omega \times \omega^\omega) - Q$. Then A^* is Σ_1^1 and its vertical sections, being cofinite, are of measure one under any continuous probability measure on ω^ω , are comeager and are dense G_δ 's. But, as is easy to check, A^* does not admit a Borel (Δ_1^1) uniformization. This shows that the uniformization results of Blackwell and Ryll-Nardzewski [1], Sarbadhikari [3] and Srivastava [4] do not extend to analytic sets.

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