A GENERALIZED PRESSING-DOWN LEMMA AND ISOCOMPACTNESS

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ABSTRACT. A topological generalization of the Pressing-Down Lemma (= Neumer's Theorem) is formulated and proved. It is shown that this result implies several recent results on isocompactness due to Worrell and Wicke.

Introduction. Let κ be an uncountable regular cardinal. A set $K \subseteq \kappa$ is a *cub* (= *closed unbounded*) set in κ if K is cofinal in κ and closed in the order topology on κ . A set $S \subseteq \kappa$ is *stationary* in κ iff $S \cap K \neq \emptyset$ for every cub set K in κ . If $A \subseteq \kappa$, a function $f: A \rightarrow \kappa$ is a *pressing-down* (or *regressive*) function iff $f(\alpha) < \alpha$ for each $\alpha \in A$. One of the most useful elementary results in set theory is the

PRESSING-DOWN LEMMA. *If S is a stationary subset of the uncountable regular cardinal κ , and $f: S \rightarrow \kappa$ is a pressing-down function, then there are an $\eta \in \kappa$ and a $C \in [S]^\kappa$ such that $f(\alpha) < \eta$ for all $\alpha \in C$. ($[S]^\kappa = \{A \subseteq S: |A| = \kappa\}$.)*

In this form the Pressing-Down Lemma (PDL) is also known as Neumer's Theorem [Ne]. A stronger version, PDL^+ , is due to Fodor [Fo]; it asserts that the set C can be chosen to be a stationary set. (Actually, both the PDL and the PDL^+ are frequently stated in a slightly different form in which ' $f(\alpha) < \eta$ ' is replaced by ' $f(\alpha) = \eta$ '. This is only an apparent strengthening, since the union of fewer than κ bounded (nonstationary, resp.) subsets of κ is bounded (nonstationary, resp.), and the present formulation is more apt for the sequel.)

By generalizing the notion of a cofinal set, Juhász [Ju] has found one purely topological generalization of the PDL. In this note I generalize the notion of a cub set to give another topological version of the PDL, one which implies some recent results of Worrell and Wicke on isocompactness and weak covering properties [WoWi₂]. (However, the proof of (i) of the Main Theorem below is very similar to the argument in [Ju].)

Definitions. Let X be a topological space. If $\mathcal{F}$ is a free closed filter on X , let $k(\mathcal{F}) = \inf\{\kappa > \omega: \exists \mathcal{Q} \in [\mathcal{F}]^\kappa (\bigcap \mathcal{Q} = \emptyset)\}$, and let $\bar{\mathcal{F}} = \{\bigcap \mathcal{Q}: \mathcal{Q} \in [\mathcal{F}]^{<k(\mathcal{F})}\}$. (As usual, $[A]^{<\kappa}$ denotes $\{B \subseteq A: |B| < \kappa\}$.) The following observations are easily verified.

- (1) $\bar{\mathcal{F}}$ is a free closed filter on X .
- (2) $k(\bar{\mathcal{F}}) = \text{cf } k(\mathcal{F})$.

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(3) $\bar{\mathcal{F}}$ is $k(\bar{\mathcal{F}})$ -complete (= closed under intersections of fewer than $k(\bar{\mathcal{F}})$ of its elements).

(4) $\bar{\bar{\mathcal{F}}} = \bar{\mathcal{F}}$.

Say that $\mathcal{F}$ is *regular* iff $\mathcal{F} = \bar{\mathcal{F}}$.

(5) Every free closed ultrafilter on X is regular.

(6) If $\mathcal{F}$ is regular, then $k(\mathcal{F})$ is a regular cardinal. (For cf $k(\mathcal{F}) = k(\bar{\mathcal{F}}) = k(\mathcal{F})$.)

(7) If $\kappa > \omega$ is a regular cardinal, then $\mathcal{F} = \{K \subseteq \kappa: K \text{ is a cub set in } \kappa\}$ is a (regular) free closed ultrafilter on κ , and $k(\mathcal{F}) = \kappa$.

Observation (7) provides a basis for generalizing the PDL.

Now fix a regular free closed filter, $\mathcal{F}$, on X . A set $S \subseteq X$ is $\mathcal{F}$ -stationary iff $\forall F \in \mathcal{F} (S \cap F \neq \emptyset)$; otherwise, S is $\mathcal{F}$ -nonstationary. $\mathcal{F}^* = \{A \subseteq X: \exists F \in \mathcal{F} (F \subseteq A)\}$, the filter (on $\mathcal{P}(X)$) generated by $\mathcal{F}$. A *neighborset* on a set $A \subseteq X$ is a function $V: A \rightarrow \tau X$ (= the topology on X) such that $x \in V(x)$ for all $x \in A$; V is $\mathcal{F}$ -bounded iff $V(x)$ is $\mathcal{F}$ -nonstationary for all $x \in A$, and V is *pseudoconstant at x on C* (where $C \subseteq A$) iff $x \in \bigcap V[C]$.

(8) If $\kappa > \omega$ is a regular cardinal, $\mathcal{F} = \{K \subseteq \kappa: K \text{ is a cub set in } \kappa\}$, $A \subseteq \kappa$, and $f: A \rightarrow \kappa$ is a pressing-down function, then the neighborset $V: A \rightarrow \tau\kappa: \alpha \mapsto (f(\alpha), \alpha]$ is $\mathcal{F}$ -bounded.

Finally, $\Delta^*(X) = \inf\{\kappa > \omega: X \text{ has no closed, discrete subset of power } \kappa\}$, and $s^*(X) = \inf\{\kappa > \omega: X \text{ has no (relatively) discrete subset of power } \kappa\}$; obviously $\Delta(X) < \Delta^*(X) < s^*(X) > s^*(X)$, where $s(X)$ and $\Delta(X)$ are, respectively, the spread and the closed discreteness character of X . And if $x \in X$ and $\mathcal{C} \subseteq \mathcal{P}(X)$, then $\text{st}(x, \mathcal{C}) = \bigcup \{C \in \mathcal{C}: x \in C\}$.

Results. The following propositions have straightforward proofs.

PROPOSITION 0. *If $S \subseteq X$ is $\mathcal{F}$ -stationary, where $\mathcal{F}$ is a regular free closed filter on X , then S is not the union of fewer than $k(\mathcal{F})$ $\mathcal{F}$ -nonstationary subsets of X . $\square$*

PROPOSITION 1. *If $W \in \tau X$ is $\mathcal{F}$ -stationary, where $\mathcal{F}$ is a free closed ultrafilter on X , then $W \in \mathcal{F}^*$. $\square$*

MAIN THEOREM. *Let $S \subseteq X$ be $\mathcal{F}$ -stationary, where $\mathcal{F}$ is a regular free closed filter on X , and let $V: S \rightarrow \tau X$ be an $\mathcal{F}$ -bounded neighborset. Assume further that $k(\mathcal{F}) > \Delta^*(X)$.*

(i) *If $k(\mathcal{F}) > s^*(X)$, or $\mathcal{F}$ is a closed ultrafilter, then there is an $x \in S$ such that $\text{st}(x, \text{ran } V)$ is $\mathcal{F}$ -stationary (and hence, in the latter case, an element of $\mathcal{F}^*$ by Proposition 1).*

(ii) *If $k(\mathcal{F}) > s^*(X)$, or $\mathcal{F}$ is a closed ultrafilter, then there are an $x \in S$ and a $C \in [S]^{k(\mathcal{F})}$ such that V is pseudoconstant at x on C .*

(iii) *If $k(\mathcal{F}) > s^*(X)$, then there are an $x \in S$ and an $\mathcal{F}$ -stationary $T \subseteq S$ such that V is pseudoconstant at x on T .*

$\Delta^*(\kappa) = \omega$ for any regular cardinal $\kappa > \omega$, so the PDL is an immediate consequence of (ii) and observations (7) and (8).

PROOF OF MAIN THEOREM. Note first that (i) implies (ii). Let x be as in (i), so that $\text{st}(x, \text{ran } V)$ is $\mathcal{F}$ -stationary, and let $C = \{y \in S: x \in V(y)\}$. Let $W = \bigcup V[C]$;

then in fact $W = \text{st}(x, \text{ran } V)$, so that W is $\mathcal{F}$ -stationary. And W is the union of $|C|$ $\mathcal{F}$ -nonstationary sets, so—by Proposition 0 — $|C| > k(\mathcal{F})$.

We now prove (i). Let $W = \bigcup \text{ran } V$, so that $S \subseteq W \in \tau X$. If $\mathcal{F}$ is a closed ultrafilter, then by Proposition 1, $W \in \mathcal{F}^*$, so there is an $F_0 \in \mathcal{F}$ such that $F_0 \subseteq W$; let $A = S \cap F_0$. Otherwise, let $A = S$. (In either case A is $\mathcal{F}$ -stationary.) Let $D \subseteq A$ be maximal with respect to the following property: if $x, y \in D$, and $x \neq y$, then $x \notin \text{st}(y, \text{ran } V)$. Clearly D is discrete, so $|D| < s^*(X)$. Moreover, D is closed in W ; thus, if $\mathcal{F}$ is a closed ultrafilter, then D is closed in F_0 (and hence in X), so in this case we even have that $|D| < \Delta^*(X)$. In either case, $|D| < k(\mathcal{F})$. But $\text{st}(D, \text{ran } V) (= \bigcup \{\text{st}(x, \text{ran } V) : x \in D\})$, since it contains A (by the maximality of D), is $\mathcal{F}$ -stationary, so, by Proposition 0, $\text{st}(x, \text{ran } V)$ is $\mathcal{F}$ -stationary for some $x \in D$.

To prove (iii), suppose that V is not pseudoconstant on any $\mathcal{F}$ -stationary $T \subseteq S$, and let $\lambda = k(\mathcal{F})$. Suppose further that $\alpha < \lambda$ and that we have already chosen points $x_\xi \in S$ (for $\xi < \alpha$) such that for each $\eta < \alpha$, $\{x_\xi : \xi < \alpha\} \cap V(x_\eta) = \{x_\eta\}$. Let $A = \{x_\xi : \xi < \alpha\}$; $|A| < \lambda$, so $V[A]$ is $\mathcal{F}$ -nonstationary. Fix $F_0 \in \mathcal{F}$ disjoint from $V[A]$. Moreover, $\mathcal{F}$ is λ -complete, and V is not pseudoconstant on an $\mathcal{F}$ -stationary $T \subseteq S$ at any $x \in A$, so there is an $F_1 \in \mathcal{F}$ such that $F_1 \subseteq F_0$ and $V[F_1 \cap S] \cap A = \emptyset$. Pick $x_\alpha \in F_1 \cap S$. Then $x_\alpha \notin V[A]$, and $V(x_\alpha) \cap A = \emptyset$.

Thus, the recursion goes through to λ , and there is a $D_0 \in [S]^\lambda$ such that $V(x) \cap D_0 = \{x\}$ for each $x \in D_0$. Clearly D_0 is discrete, which is impossible, since $s^*(X) \leq \lambda = |D_0|$. This completes the proof. $\square$

Part (iii) of the theorem is the best approximation I could get to the PDL^+ , but it does not imply the PDL^+ , since $s^*(\kappa) = \kappa^+$ for any cardinal κ . (Juhász was also unable to extend his generalization to include the PDL^+ .) I suspect that the difficulty is inherent in the nature of the PDL^+ , which seems to require some notion of diagonal intersection for its proof: diagonal intersections are heavily dependent upon considerations of order and cardinality.

An application. A popular question in recent years has been: How weak can a covering property, P , be and still satisfy the equation ‘countably compact + P = compact’? The excellent solution ‘ P = weakly $\delta\theta$ -refinable’ [**WiWo**₀] is notorious for having been overlooked for ten years after its first mention in [**WoWi**₁]. Even it, however, is superseded by the results in [**WoWi**₂], of which it is an easy corollary. Here I show that (a strengthening of) the main theorem of [**WoWi**₂] is itself a corollary of the above generalized PDL .

If X is a space, κ is a cardinal, and $\mathcal{U}$ is an open cover of X , a *weak* $[\kappa]$ -refinement of $\mathcal{U}$ is an open refinement, $\mathcal{R} = \bigcup \{\mathcal{R}_\alpha : \alpha \in \lambda\}$, of $\mathcal{U}$ such that: (a) $\lambda < \kappa$, and (b) for each $x \in X$ there is an $\alpha \in \lambda$ such that $0 < |\{R \in \mathcal{R}_\alpha : x \in R\}| < \kappa$. The main theorem of [**WoWi**₂] (slightly modified) can now be stated as follows.

COROLLARY 1 [**WoWi**₂]. *Let X be a T_1 -space such that every well-ordered, increasing open cover, $\mathcal{U}$, of X of regular, uncountable cardinality has a weak $[|\mathcal{U}|]$ -refinement. If $\mathcal{F}$ is a free closed ultrafilter on X , then $k(\mathcal{F}) < \Delta^*(X)$. $\square$*

In fact, however, a stronger result is true. In the definition given above of a weak $[\kappa]$ -refinement of $\mathcal{U}$, replace condition (b) by the following weaker condition (b'): for each $x \in X$ there are an $\alpha \in \kappa$ and a $\mathcal{V} \in [\mathcal{U}]^{<\kappa}$ such that $\emptyset \neq \text{st}(x, \mathcal{R}_\alpha) \subseteq \bigcup \mathcal{V}$. An $\mathcal{R}$ satisfying (a) and (b') will be called a *very weak* $[\kappa]$ -refinement of $\mathcal{U}$.

COROLLARY 1'. *Let X be a T_1 -space such that every well-ordered, increasing open cover, $\mathcal{U}$, of X of regular uncountable cardinality has a very weak $[[\mathcal{U}]]$ -refinement. If $\mathcal{F}$ is a free closed ultrafilter on X , then $k(\mathcal{F}) < \Delta^*(X)$.*

PROOF. Suppose that $\mathcal{F}$ is a free closed ultrafilter on X for which $k(\mathcal{F}) > \Delta^*(X)$, and let $\lambda = k(\mathcal{F})$. Clearly $\lambda > \omega$ unless $\lambda = \Delta^*(X) = \omega$; but $\Delta^*(X) = \omega$ iff X is countably compact, in which case $\mathcal{F}$ is at least ω_1 -complete, so in all cases $\lambda > \omega$. And of course λ is regular.

Let $\mathcal{F}_0 = \{F_\alpha : \alpha \in \lambda\} \subseteq \mathcal{F}$ be such that $\bigcap \mathcal{F}_0 = \emptyset$ and $F_\alpha \supset F_\beta$ whenever $\alpha < \beta < \lambda$, and, for $\alpha \in \lambda$, let $U_\alpha = X \setminus F_\alpha$. Let $\mathcal{R} = \bigcup \{\mathcal{R}_\alpha : \alpha \in \kappa\}$ be a very weak $[\lambda]$ -refinement of $\{U_\alpha : \alpha \in \lambda\}$ (as described above) for some $\kappa < \lambda$. For each $\alpha \in \kappa$, let $A_\alpha = \{x \in X : \emptyset \neq \text{st}(x, \mathcal{R}_\alpha) \subseteq \bigcup \{U_\alpha : \alpha < \mu\} \text{ for some } \mu < \lambda\}$. Then $X = \bigcup \{A_\alpha : \alpha \in \kappa\}$, so, by Proposition 0, there is an $\eta \in \kappa$ such that A_η is $\mathcal{F}$ -stationary. Let $N: A_\eta \rightarrow \mathcal{R}_\eta$ be a neighbornet; there is then, by the Main Theorem, an $x \in A_\eta$ for which $\text{st}(x, \text{ran } N)$ is $\mathcal{F}$ -stationary. But then $\text{st}(x, \text{ran } N) \subseteq \text{st}(x, \mathcal{R}_\alpha) \subseteq \bigcup \{U_\alpha : \alpha < \mu\}$ for some $\mu < \lambda$, so, again by Proposition 0, some U_α ($\alpha < \mu$) is $\mathcal{F}$ -stationary—which is absurd. Thus, no such $\mathcal{F}$ exists. $\square$

A similar argument yields the following results, given the observation that $s^*(X) > \omega$ for any infinite Hausdorff space X .

COROLLARY 2. *Let X be as in Corollary 1', but assume further that X is an infinite Hausdorff space. If $\mathcal{F}$ is a regular free closed filter on X , then $k(\mathcal{F}) < s^*(X)$. $\square$*

COROLLARY 3. *Let X be as in Corollary 2, and let $\mathcal{F}$ be a free closed filter on X . Then $\text{cf}(\mathcal{F}) < s^*(X)$.*

PROOF. Apply Corollary 2 to $\bar{\mathcal{F}}$. $\square$

REMARK. No separation axioms are required in the above corollaries if the conclusions are weakened to ' $k(\mathcal{F}) < \Delta(X)$ ', ' $k(\mathcal{F}) < s(X)$ ', and ' $\text{cf } k(\mathcal{F}) < s(X)$ ', respectively. Moreover, Corollaries 2 and 3 almost apply to T_1 -spaces. First, it is clearly sufficient to assume that X has an infinite Hausdorff subspace, for that is equivalent to $s^*(X) > \omega$. And it can be shown fairly easily that a T_1 -space has no infinite Hausdorff subspace iff it is hereditarily compact, in which case the magnitude of $k(\mathcal{F})$ is unlikely to be of much interest.

COROLLARY 4. *If X is as in Corollary 1', then X is linearly $(\infty, s(X)^+)$ -compact, i.e., every linearly ordered (by inclusion) open cover of X has a subcover of cardinality at most $s(X)$. $\square$*

REFERENCES

- [Fo] G. Fodor, *Eine Bemerkung zur Theorie der regressiven Funktionen*, Acta Sci. Math. (Szeged) **17** (1956), 139–142.
- [Ju] I. Juhász, *On Neumer's Theorem*, Proc. Amer. Math. Soc. **54** (1976), 453–454.
- [Ne] W. Neumer, *Verallgemeinerung eines Satzes von Alexandroff und Urysohn*, Math. Z. **54** (1951), 254–261.
- [WiWo] H. H. Wicke and J. M. Worrell, Jr., *Point-countability and compactness*, Proc. Amer. Math. Soc. **55** (1976), 427–431.
- [WoWi₁] J. M. Worrell, Jr. and H. H. Wicke, *Characterizations of developable topological spaces*, Canad. J. Math. **17** (1965), 820–830.
- [WoWi₂] ———, *A covering property which implies isocompactness*. I (preprint).

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