FIXED POINT SETS OF LC^∞ , C^∞ CONTINUA

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ABSTRACT. A space X is said to have the complete invariance property (CIP) if every nonempty closed subset of X is the fixed point set of some self-map of X . An example is given to show that there is an LC^∞ , C^∞ continuum X which does not have CIP. Moreover, X is a wedge of two LC^∞ , C^∞ continua each of which has CIP.

1. Introduction. A subset A of a space X is called a *fixed point set* of X if there is a continuous function $f: X \rightarrow X$ such that $f(x) = x$ iff $x \in A$. In [8, p. 553] L. E. Ward, Jr. defines a space X to have the *complete invariance property* (CIP) if every nonempty closed subset of X is a fixed point set of X . Some spaces known to have CIP include spaces which admit a strongly convex metrization [8, p. 554], compact manifolds without boundary [6, p. 378], a class of polyhedra which includes all compact triangulable manifolds (with or without boundary) [7, p. 225], locally compact metric groups [4, 5.4], and all 1-dimensional Peano continua [5]. Examples of finite-dimensional continua possessing higher order local connectedness but which fail to have CIP have been constructed in [3] and [4, 3.2].

The purpose of this paper is to show that there is an LC^∞ , C^∞ continuum X which does not have CIP. In addition, it is shown that X is a wedge $Z \vee Z$ (that is, the disjoint union of two copies of Z with a single point in one copy of Z identified with a single point in the other copy of Z) of an LC^∞ , C^∞ continuum Z with itself such that Z has CIP.

2. Preliminaries and the construction of X . The terminology used in this paper may be found in [1]. In particular, Hilbert space E^∞ with metric ρ and Euclidean n -dimensional space E^n are as defined in [1, pp. 10–11]. Henceforth, we follow [2, pp. 36–37] and a point $x = (x_1, x_2, x_3, \dots, x_n, 0, \dots) \in E^n$ will be denoted by its cylindrical coordinates $(r, \theta, x_3, \dots, x_n, 0, \dots)$ where $r = (x_1^2 + x_2^2)^{1/2}$ and $x_1 = r \cos \theta$, $x_2 = r \sin \theta$. Let D , S^1 denote the unit disk and the unit circle in E^2 defined by

$$D = \{x \in E^2 \mid r < 1\} \quad \text{and} \quad S^1 = \{x \in E^2 \mid r = 1\}.$$

Let $c = (2, 0, \dots) \in E^1$ and let Z^n denote the n -sphere in E^{n+2} defined by

$$Z^n = \{x \in E^{n+2} \mid \theta = 0 \text{ and } \rho(x, c) = 1\}.$$

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If $-\infty < t < \infty$, define $d_t: E^\omega \rightarrow E^\omega$ by

$$d_t(x) = \begin{cases} (r, \theta + t\pi, 2^{-t}x_3, 2^{-t}x_4, \dots) & \text{if } r \leq 1, \\ (1 + (r-1)2^{-t}, \theta + t\pi, 2^{-t}x_3, 2^{-t}x_4, \dots) & \text{if } r > 1. \end{cases}$$

Let $Z_t^n = d_t(Z^n)$. Then Z_t^n is an n -sphere of radius 2^{-t} such that $Z_t^n \cap D = (1, t\pi, 0, \dots) \in S^1$ and Z_s^n is tangent to Z_t^n if $s \equiv t \pmod{2}$. If J is an interval in the real line, let $Z_J^n = \bigcup_{t \in J} Z_t^n$.

DEFINITIONS. (1) We define $Z = D \cup \bigcup_{n=1}^{\infty} Z_{[(n-1)\pi, n(n+1)\pi]}^n$.

(2) We define X to be the wedge of Z with itself at the point $b = (1, \pi, 0, \dots) \in S^1$.

In the sequel we shall make use of the following basic result due to L. E. Ward, Jr. [8, p. 554].

WARD'S LEMMA. *Let X be a metric space and let $p \in X$. Suppose there exists a homotopy $H: X \times I \rightarrow X$ such that*

- (1) $H(x, 0) = x$ for all $x \in X$,
- (2) $H(x, t) \neq x$ if $x \neq p$ and $t > 0$.

If A is a closed subset of X containing p , then A is a fixed point set of X .

3. The properties of X and Z .

PROPERTY 1. *Each of X and Z has the property LC^∞ .*

PROOF. We give a proof for Z and remark that the case for X readily follows. If $p \in Z - S^1$, then p has a neighborhood which is finitely triangulable, and therefore Z is locally contractible at p . If $p \in S^1$, then p has a neighborhood which has the homotopy type of $\bigcup_{n=0}^{\infty} Z_{2n}^{[n]}$ where $[n] = k$ if $(k-1)k \leq 2n < k(k+1)$, $k = 1, 2, \dots$. It follows that Z is LC^∞ .

PROPERTY 2. *Each of X and Z has the property C^∞ .*

PROOF. We give a proof for Z and remark that this proof can easily be modified to include the case for X . Clearly, Z is an arcwise connected continuum and, for each $n = 1, 2, \dots$, $D \cup \bigcup_{k=n+1}^{\infty} Z_{[(k-1)k, k(k+1)]}^k$ is a deformation retract of Z which is C^n . It follows that Z is C^∞ .

PROPERTY 3. *The space Z has CIP.*

PROOF. Let A be a nonempty closed subset of Z . To complete the proof there are three cases to consider.

Case 1. Suppose $A \subset Z - D$. Since A is compact, there is a finite sequence of cells $C_1, C_2, \dots, C_n$ in $Z - D$ such that each C_i is an $(i+1)$ -cell, $C_i \cap C_{i+1}$ is a $(i-1)$ -cell, $C_i \cap C_j = \emptyset$ if $j \neq i-1, i+1$, and $A \subset C$ where $C = \bigcup_{i=1}^n C_i$. Moreover, C is a polyhedron of type W (see [7, p. 222]) and hence has CIP [7, p. 224]. Since C is a retract of Z , it follows that A is a fixed point set of Z .

Case 2. Suppose $A \cap (D - S^1) \neq \emptyset$. Without loss of generality, we shall assume that the origin 0 lies in A . Let $H: Z \times I \rightarrow Z$ be a homotopy defined by $H(x, t) = d_t(x)$ if $x \in Z$, $t \in I$. Then, by Ward's Lemma, A is a fixed point set of Z .

Case 3. Suppose $p \in A \cap D \subset A \cap S^1$. Let $q = (1, 0, \dots) \in E^1$. We observe that, if we regard $Z_0^n - \{q\}$ as Euclidean n -dimensional space, a translation of $Z_0^n - \{q\}$ leaves the point q at infinity fixed. It then follows that there is a sequence of homotopies $F_n: Z_0^n \times I \rightarrow Z_0^n$, $n = 1, 2, \dots$, such that the following properties are satisfied.

- (1) $F_n|_{Z_0^n \times \{t\}}$ is a homeomorphism for each $t \in I$.
- (2) $F_n(x, 0) = x$ for all $x \in Z_0^n$.
- (3) $F_n(q, t) = q$ for all $t \in I$.
- (4) $F_n(x, t) \neq x$ if $t > 0$ and $x \neq q$.
- (5) $F_n|_{Z_0^{n-1} \times I} = F_{n-1}$ for $n = 2, 3, \dots$.

Similarly, let $G: D \times I \rightarrow D$ be a homotopy such that the following properties are satisfied.

- (1) For each $t \in I$, $G|_{D \times \{t\}}$ is a homeomorphism which maps each circle in D tangent to p onto itself by an orientation-preserving homeomorphism.
- (2) $G(x, 0) = x$ for all $x \in D$.
- (3) $G(p, t) = p$ for all $t \in I$.
- (4) $G(x, t) \neq x$ if $t > 0$ and $x \neq p$.

For $t \in I$, $s > 0$, let $\theta(s, t)$ denote the unique number such that $0 < \theta(s, t) < 2$ and $G((1, s\pi, 0, \dots), t) = (1, (s + \theta(s, t))\pi, 0, \dots)$.

Define a homotopy $H: Z \times I \rightarrow Z$ by

$$H(x, t) = \begin{cases} G(x, t) & \text{if } x \in D, t \in I, \\ d_{s+\theta(s,t)}(F_n(d_{-s}(x), t)) & \text{if } x \in Z_s^n, t \in I. \end{cases}$$

It is easy to check that H is a homotopy satisfying the hypothesis of Ward's Lemma, and therefore A is a fixed point set of Z .

PROPERTY 4. *The space X does not have CIP.*

PROOF. Let $h: Z \rightarrow E^\omega$ be a homeomorphism from Z into E^ω such that $Z \cap h(Z) = \emptyset$. Then X may be regarded as the continuum obtained by taking the union of Z and $h(Z)$, and then identifying the point $b = (1, \pi, 0, \dots) \in Z$ with the point $h(b) \in h(Z)$. Let A denote the closed subset of X defined by

$$A = \bigcup_{n=1}^{\infty} Z_{(n-1)\pi}^n \cup h\left(\bigcup_{n=1}^{\infty} Z_{(n-1)\pi}^n\right).$$

Suppose there is a continuous function $f: X \rightarrow X$ whose fixed point set is A . Since $f(b) \neq b$, either $f(b) \in Z$ or $f(b) \in h(Z)$. Without loss of generality, we shall assume that $f(b) \in h(Z)$. Let $r: X \rightarrow Z$ be a retraction of X onto Z defined by

$$r(x) = \begin{cases} x & \text{if } x \in Z, \\ b & \text{if } x \in h(Z). \end{cases}$$

It follows that for some positive integer k , $rf(\bigcup_{n=k}^{\infty} Z_{(n-1)\pi}^n) = b$. Define a homotopy $H: (\bigcup_{n=k}^{\infty} Z_{(n-1)\pi}^n) \times I \rightarrow Z$ by

$$H(x, t) = rf d_t(x) \quad \text{if } x \in \bigcup_{n=k}^{\infty} Z_{(n-1)\pi}^n, t \in I.$$

Then it is easy to check that H is a deformation with values in Z which deforms $\bigcup_{n=k}^{\infty} Z_{(n-1)n}^n$ to the point b . This is a contradiction since any such deformation with values in Z which deforms $Z_{(n-1)n}^n$ to a point must trace out at least one path of length $\geq 2\pi n$. Consequently, A cannot be a fixed point set of X and, therefore, X does not have CIP.

We remark that the space $B^n = D \cup Z_{[0,\infty)}^n$ is an $(n+1)$ -dimensional acyclic LC^{n-1} continuum which was constructed by R. J. Knill in [2, p. 37]. The results of our paper can easily be modified to show that B^n has CIP and that the wedge of B^n with itself at the point b fails to have CIP.

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