THE STATES ASSOCIATED WITH APPROXIMATELY INNER AUTOMORPHISMS

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ABSTRACT. Let M be a II_1 -factor acting standardly on a Hilbert space H . For an approximately inner automorphism θ of M , there exists a state φ on $B(H)$ associated with θ . If the symmetry σ of $M \otimes M$ is approximately inner on $M \otimes M$, then, by restricting the state associated with σ to $B(H) \otimes I$, we have a hypertrace of M .

1. Introduction. Let M be a factor of type II_1 with the canonical trace τ , acting standardly in the Hilbert space $H = L^2(M, \tau)$. In [1], A. Connes proved that the following three statements are equivalent; (1) M is hyperfinite, (2) the automorphism σ on $M \otimes M$ (which is called the symmetry [2]) such that $\sigma(x \otimes y) = y \otimes x$ ($x \in M, y \in M$) is approximately inner on $M \otimes M$, and (3) there exists a state φ on $B(H)$ (which is called a hypertrace of M) such that $\varphi(ax) = \varphi(xa)$ ($x \in B(H), a \in M$). Following the order of $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$, he showed this result by passing through several other properties. As a consequence, if the symmetry σ is approximately inner on $M \otimes M$, then there exists a hypertrace φ of M , but it is not clear the relation between the symmetry σ and the hypertrace φ .

In this paper, we shall show that, for an approximately inner automorphism θ on M , there exists a state φ on $B(H)$ associated with θ . And we shall show that if the symmetry σ is approximately inner on $M \otimes M$, then the restriction of a state associated with σ to $B(H) \otimes I$ is a hypertrace of M .

2. Results. Denote by ζ the canonical imbedding of M into H and by J the canonical involution of the cyclic and separating vector $\zeta(1)$.

THEOREM 1. *Let θ be an approximately inner automorphism of M ; then there exists a state φ on $B(H)$ such that $\varphi(\theta(u)JuJ) = 1$ for every unitary operator u in M .*

PROOF. Since θ is approximately inner on M , by [1, Theorem 3.1], we have a bounded sequence $(x_n)_{n \in \mathbb{N}}$ in M , not converging strongly to 0, such that $x_n a - \theta(a)x_n$ converges to 0 strongly for any a in M . Hence, there exists a positive number α and a subsequence $(y_n)_{n \in \mathbb{N}}$ of the sequence $(x_n)_{n \in \mathbb{N}}$ such that $\|y_n\|_2 > \alpha$. For each integer n , put $\xi_n = \zeta(y_n)/\|y_n\|_2$ and let w_n be the vector state on $B(H)$ defined by ξ_n . Take a weak limit point φ of the set $\{w_n; n = 1, 2, \dots\}$. Then φ is a desired state on $B(H)$, because, for every unitary $u \in M$, we have that $\|\xi_n - JuJ\theta(u)\xi_n\| = \|y_n u - \theta(u)y_n\|_2/\|y_n\|_2$.

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We shall call φ in Theorem 1 a state associated with θ . A state φ associated with an approximately inner automorphism θ is not necessarily unique for θ in general. But if θ is inner on M , then φ is unique in the sense of the following. If θ is an inner automorphism on M induced by a unitary u in M , then we can take φ as the vector state defined by u . Conversely if φ is the vector state defined by a unitary u in M , then θ is an inner automorphism induced by u . Specifically, θ is the identity on M if and only if φ is the trivial state (i.e. the vector state defined by the identity operator). In fact, if θ is inner on M , then by the proof of Theorem 1, φ becomes such as the state as above. Conversely, if there exists a unitary v in M such that $\varphi(x) = (x\zeta(v), \zeta(v))$ for every x in $B(H)$, then for every unitary u in M , we have that

$$\|\theta(u)vu^* - v\|_2^2 = (\zeta(\theta(u)vu^* - v), \zeta(\theta(u)vu^* - v)) = 2 - 2\varphi(\theta(u)JuJ) = 0.$$

Since θ is an automorphism of M , it follows that $\theta(a) = vav^*$ for all $a \in M$, so that θ is an inner automorphism induced by v .

COROLLARY 2. *Let φ be a state associated with an approximately inner automorphism θ of M ; then φ is an extension of the canonical trace τ of M .*

PROOF. By Schwartz's inequality and the property of φ , we have that

$$\varphi(JuJ\theta(u)x) = \varphi(x) = \varphi(xJuJ\theta(u)) \quad (x \in B(H), \text{ unitary } u \in M). \quad (*)$$

Hence, $\varphi(a) = \varphi(\theta(u)a\theta(u^*))$ for all a in M and all unitary u in M . Since θ is an automorphism of M , then the restriction of φ to M is the canonical trace of M .

COROLLARY 3. *If the symmetry σ of $M \otimes M$ is approximately inner on $M \otimes M$, then a state φ associated with σ is an extension of a hypertrace of M .*

PROOF. By the equality (*), for all x in $B(H)$ and all unitary u in M , we have that

$$\begin{aligned} \varphi(ux \otimes 1) &= \varphi((1 \otimes JuJ)(u \otimes 1)(x \otimes Ju^*J)) \\ &= \varphi((x \otimes Ju^*J)(1 \otimes JuJ)(u \otimes 1)) = \varphi(xu \otimes 1), \end{aligned}$$

so that, $B(H) \ni x \rightarrow \varphi(x \otimes 1)$ is a hypertrace of M .

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