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# RINGS WITH EVERY PROPER IMAGE A PRINCIPAL IDEAL RING

P. F. SMITH

ABSTRACT. The main result of this paper states that if $R$ is a right Noetherian right bounded prime ring such that nonzero prime ideals are maximal and such that every proper homomorphic image of $R$ is a principal right ideal ring then $R$ is right hereditary.

In [**10**, Theorem 8] it is proved that if $R$ is a right bounded prime ring of finite right Goldie dimension such that every proper homomorphic image is a right Artinian principal right ideal ring then $R$ is right hereditary. It is not difficult to see that such rings are in fact right Noetherian and so [**10**, Theorem 8] is a consequence of the main result of this present paper. In fact, it is shown here that if $R$ is a right Noetherian right bounded prime ring, such that nonzero prime ideals are maximal and such that for all nonzero prime ideals $P$, $Q$ (not necessarily distinct) the ring $R/PQ$ is a principal right ideal ring, then $R$ is right hereditary (compare [**10**, Theorem 9]).

In a recent paper, Hajarnavis and Norton [**5**, Theorem 6.4] proved that if $R$ is a (right and left) Noetherian right bounded prime ring whose proper homomorphic images are ipri-rings then $R$ is a Dedekind prime ring. The proof involves localization at every nonzero prime ideal of the ring $R$. We show how to deduce this result and in so doing completely remove localization techniques from the proof, making it much more elementary.

If $I$ is any ideal of a ring $R$ then $\mathcal{C}(I)$ will denote the set of elements $c$ in $R$ such that $c + I$ is a regular element of the ring $R/I$. The ring $R$ is an *ipri-ring* if every (two-sided) ideal is principal as a right ideal.

LEMMA 1. *Let $R$ be a right Noetherian prime ring such that every proper homomorphic image is an ipri-ring. If $M_1$ and $M_2$ are distinct maximal ideals of $R$ then $M_1M_2 = M_2M_1 = M_1 \cap M_2$.*

PROOF. Since $R$ is prime, but not simple, it follows that $M_1M_2 \cap M_2M_1 \neq 0$. Let $\bar{R} = R/(M_1M_2 \cap M_2M_1)$. Then $\bar{R}$ is a right Noetherian ipri-ring and $\overline{M_1}$ and $\overline{M_2}$ are distinct maximal ideals of $\bar{R}$, where $^-$ denotes images in $\bar{R}$. There exist elements $c_1$, $c_2$ in $R$ such that $\overline{M_1} = \bar{c}_1\bar{R}$ and $\overline{M_2} = \bar{c}_2\bar{R}$. By [**2**, Theorem 3.9], $c_1 \in \mathcal{C}(M_2)$ and $c_2 \in \mathcal{C}(M_1)$. It follows that $\overline{M_1} \cap \overline{M_2} \subseteq \bar{c}_1\overline{M_2} = \overline{M_1}\,\overline{M_2}$ and hence $M_1 \cap M_2 \subseteq M_1M_2$. Thus $M_1 \cap M_2 = M_1M_2$ and similarly $M_1 \cap M_2 = M_2M_1$.

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An ideal $I$ of a ring $R$ has the *right AR property* if for each right ideal $E$ there exists a positive integer $n$ such that $E \cap I^n \subseteq EI$.

LEMMA 2 (SEE [7, THEOREM 1.2]). *Let $R$ be a right Noetherian prime ring and $I$ a proper ideal such that $I$ has the right AR property. Then* $\bigcap_{n=1}^{\infty} I^n = 0$.

If $E$ is a right ideal of a ring $R$ then the *bound* of $E$ is the unique largest (two-sided) ideal contained in $E$. The ring $R$ is *right bounded* if every essential right ideal has nonzero bound.

LEMMA 3. *Let $R$ be a right Noetherian right bounded prime ring such that every proper homomorphic image is an ipri-ring. Then the following statements are equivalent.*

(i) *Nonzero prime ideals of $R$ are maximal ideals.*

(ii) *Every ideal of $R$ has the right AR property.*

PROOF. (i) $\Rightarrow$ (ii) By Lemma 1 and [4, Proposition 3.2].

(ii) $\Rightarrow$ (i) Let $P \subseteq Q$ be distinct prime ideals of $R$. Suppose $P \neq 0$. Then the ring $R/P^2$ is an ipri-ring. Let $\bar{R} = R/P^2$ and let $^-$ denote images in $\bar{R}$. There exists an element $q$ in $Q$ such that $\bar{Q} = \bar{q}\bar{R}$. By [2, Theorem 3.9] $q \in \mathcal{C}(P)$ and it follows that $\bar{P} = \bar{q}\bar{P}$. Thus $P = qP + P^2 \subseteq QP$. Hence $P = QP = Q^2P = \cdots$ and $Q = R$, for otherwise $P \subseteq \bigcap_{n=1}^{\infty} Q^n = 0$ by Lemma 2.

LEMMA 4. *Let $R$ be a right Noetherian prime ring and $I$ a proper ideal of $R$ such that $I$ has the right AR property. If $E$ is an essential right ideal of $R$ and $c$ an element of $E$ such that $E = cR + EI$ then $c$ is regular.*

PROOF. Since $E = cR + EI = cR + EI^2 = \cdots$ it follows that

$$E \leqslant \bigcap_{n=1}^{\infty} (cR + I^n).$$

By [2, Theorem 3.9] $E$ contains a regular element $e$. Let $F = eR + cR$. There exists a positive integer $m$ such that $F \cap I^m \leqslant FI$. There exists $r$ in $R$ such that $e \in cr + I^m$, so that $e - cr \in FI$. Thus $e - cr = ea + cb$ for some elements $a, b$ in $I$. Thus $e(1 - a) = cs$ where $s = r + b$. By Lemma 2, $1 - a$ is regular in $R$ and hence $cs$ is regular in $R$. If $r(s)$ denotes the right annihilator of the element $s$ then $r(s) = 0$ and by [2, Lemma 3.8] $sR$ is an essential right ideal of $R$. Let $u \in R$ satisfy $cu = 0$. If $u \neq 0$ then there exists $v$, $w$ in $R$ such that $0 \neq uv = sw$. Then $cu = 0$ implies $w = 0$, a contradiction. Thus $r(c) = 0$ and it follows that $c$ is regular in $R$.

The next lemma is concerned with the question of when a right ideal which contains a projective right ideal is itself projective.

LEMMA 5. *Let $E$ be a right ideal of a ring $R$ such that $E$ contains a projective right ideal $P$. If there exists a right ideal $F$ such that $E \cap F \subseteq P$ and $E + F = R$ then $E$ is projective.*

PROOF. Clearly $R/P = (E/P) \oplus (F + P)/P$. Since $P$ is projective it follows that $R/P$ has projective dimension $\leqslant 1$. Thus $(F + P)/P$ has projective dimension $\leqslant 1$ and, because $R/E \cong (F + P)/P$, so has $R/E$. Thus $E$ is projective.

THEOREM 6. *Let $R$ be a right Noetherian right bounded prime ring such that nonzero prime ideals are maximal and every proper homomorphic image is a principal right ideal ring. Then $R$ is right hereditary.*

PROOF. By Lemma 3 every ideal of $R$ has the right AR property. Let $E$ be an essential right ideal of $R$ and let $I$ be the bound of $E$. Then $EI$ is an essential right ideal of $R$. Let $A$ be the bound of $EI$. Since $R/A$ is a principal right ideal ring there exists an element $c$ in $E$ such that $E = cR + EI$. By Lemma 4, $c$ is regular. Thus $cR$ is an essential right ideal of $R$ (see [2, Lemma 3.8]). Let $B$ be the bound of $cR$. Then $A + B \subseteq E$. Since $R$ is right Noetherian the nonzero ideal $B$ contains a product of nonzero prime, and hence maximal, ideals, say $M_1 M_2 \cdots M_n \subseteq B$ with each $M_i$ a maximal ideal of $R$. Since $A + B \subseteq E$ it follows that $A \subseteq M_i$ for some $1 \leqslant i \leqslant n$ and by Lemma 1 we can suppose without loss of generality that $i = 1$. By rearranging the ideals $M_i$ $(1 \leqslant i \leqslant n)$ using Lemma 1, if necessary, we may suppose that there exists a positive integer $m \leqslant n$ such that $I \subseteq M_i$ $(1 \leqslant i \leqslant m)$ and $I \nsubseteq M_i$ $(m + 1 \leqslant i \leqslant n)$.

Since $E = cR + EI = cR + EI^2 = \cdots$ it follows that $E \subseteq cR + I^m$. If $m = n$ then $I^n \subseteq B \subseteq cR$ and hence $E \subseteq cR$. Thus $E = cR$ and $E$ is projective. Otherwise, if $m < n$ let $J = M_{m+1} \cap \cdots \cap M_n$. Then $I^m J \subseteq cR$ and hence $EJ \subseteq cR$. There exists a positive integer $k$ such that $E \cap J^k \subseteq EJ \subseteq cR$. Moreover, $I + J = R$ implies that $I + J^k = R$ and hence $E + J^k = R$. By Lemma 5, $E$ is projective. It follows that every essential right ideal of $R$ is projective. But every right ideal of $R$ is a direct summand of an essential right ideal. Hence $R$ is right hereditary.

In what follows, by a *Noetherian ring* we shall mean a right and left Noetherian ring. If $I$ is an ideal of a ring $R$ then we call a right ideal $E$ *essential modulo $I$* if $I \subseteq E$ and $E/I$ is an essential right ideal of the ring $R/I$. The next result is well known.

LEMMA 7. *Let $M$ be a maximal ideal of a Noetherian ring $R$. Then $\mathcal{C}(M) = \mathcal{C}(M^k)$ for all positive integers $k$.*

PROOF. Suppose $\mathcal{C}(M) = \mathcal{C}(M^k)$ for some $k \geqslant 1$. We shall prove that $\mathcal{C}(M) = \mathcal{C}(M^{k+1})$. The result will then follow by induction. By a result of Djabali (see [3, Theorem 2.1]), $\mathcal{C}(M^{k+1}) \subseteq \mathcal{C}(M)$. Let

$$K = \{r \in R : rc \in M^{k+1} \text{ for some } c \text{ in } \mathcal{C}(M)\}.$$

Then $K \leqslant M^k$. By [2, Theorem 3.9] a right ideal $E$ is essential modulo $M$ if and only if $cR + M \subseteq E$ for some element $c$ in $\mathcal{C}(M)$. Thus

$$K = \{r \in R : rE \subseteq M^{k+1} \text{ for some right ideal } E \text{ essential modulo } M\}.$$

By the properties of essential right ideals of $R/M$ it follows that $K$ is an ideal of $R$. Since $R$ is left Noetherian the left ideal $K$ is finitely generated and hence $Kd \subseteq M^{k+1}$ for some $d$ in $\mathcal{C}(M)$. But $KM \subseteq M^{k+1}$ and thus

$$M \subsetneq \{r \in R : Kr \subseteq M^{k+1}\} = L.$$

Because $M$ is maximal, $L = R$ and hence $K \subseteq M^{k+1}$. Thus whenever $r \in R$, $c \in \mathcal{C}(M)$ and $rc \in M^{k+1}$ we have $r \in M^{k+1}$. By a similar argument it can be shown that if $cr \in M^{k+1}$ with $r$ in $R$ and $c$ in $\mathcal{C}(M)$ then $r \in M^{k+1}$. Thus $\mathcal{C}(M) \subseteq \mathcal{C}(M^{k+1})$ and we conclude $\mathcal{C}(M) = \mathcal{C}(M^{k+1})$.

LEMMA 8 (SEE [6, LEMMA 3.1]). *Let $I = aR$ be an ideal of a Noetherian ring $R$ such that $I \subseteq \bigcap_{n=1}^{\infty} b^n R$ for some element $b$ in $R$. Then there exists $c$ in $R$ such that $(1 - cb)I = 0$.*

LEMMA 9. *Let $R$ be a Noetherian right bounded prime ring such that every proper homomorphic image is an ipri-ring. If $P$ is a nonzero prime ideal of $R$ then the ring $R/P$ is right Artinian.*

PROOF. Let $Q \subseteq M$ be distinct prime ideals of $R$ and suppose $Q \neq 0$. Then the ring $R/Q^2$ is an ipri-ring. Let $\bar{R} = R/Q^2$ and $^-$ denote images in $\bar{R}$. As in the proof of Lemma 3 (ii) $\Rightarrow$ (i), $\bar{Q} = \bar{M}\bar{Q}$ and hence $\bar{Q} \subseteq \bigcap_{n=1}^{\infty} \bar{M}^n$. By Lemma 8, there exists an element $m$ in $M$ such that $(\bar{1} - \bar{m})\bar{Q} = \bar{0}$. By [6, Corollary 3.4], $\bar{1} - \bar{m}$ is nilpotent and hence $M = R$. Thus nonzero prime ideals of $R$ are maximal. Also by Lemma 1 maximal ideals of $R$ commute.

Let $P$ be a nonzero prime ideal of $R$. Let $c \in \mathcal{C}(P)$. By Lemma 3 $P$ has the right AR property and by Lemma 2 $\bigcap_{n=1}^{\infty} P^n = 0$. Hence by Lemma 7, $c \in \mathcal{C}(0)$. Thus $cR$ is an essential right ideal of $R$ (see [2, Theorem 3.9]) and there exist maximal ideals $M_1, \ldots, M_k$ for some positive integer $k$ such that $M_1 M_2 \cdots M_k \subseteq cR$. Suppose that $P \neq M_i$ $(1 \leqslant i \leqslant k)$. Then there exists an element $p$ in $P$ such that $1 - p \in M_1 M_2 \cdots M_k \subseteq cR$. It follows that $c + P$ is a unit in the ring $R/P$. Therefore, suppose $P = M_i$ for some $1 \leqslant i \leqslant k$. Then there exists a positive integer $t$ and an element $x$ in $P$ such that $P^t(1 - x) \subseteq cR$. In this case, $c \in \mathcal{C}(P)$ implies

$$P^t = cP^t + P^{t+1}$$

by Lemma 7. Let $R^* = R/P^{t+1}$ and let $*$ denote images in $R^*$. Since $R^*$ is an ipri-ring it follows that $(P^t)^*$ is principal as a right ideal and

$$(P^t)^* \subseteq \bigcap_{n=1}^{\infty} c^{*n} R^*.$$

By Lemma 8 there exists an element $a$ in $R$ such that

$$(1^* - a^* c^*)(P^t)^* = 0.$$

Thus $(1 - ac)P^t \subseteq P^{t+1}$. Let $K = \{r \in R : rP^t \subseteq P^{t+1}\}$. Then $K$ is an ideal of $R$ and $P \subseteq K$. If $K = R$ then $P^t = P^{t+1}$ and so $P^t = 0$ by Lemma 2. Thus $K = P$ and hence $1 - ac \in P$. It follows that $c + P$ is a unit in $R/P$. Thus, for each element $c$ in $\mathcal{C}(P)$ we have proved that $c + P$ is a unit in $R/P$. By [2, Theorems 4.1 and 4.4] the ring $R/P$ is right Artinian.

Given Lemma 9, the next result is well known but we include it and its proof for completeness.

THEOREM 10. *Let $R$ be a Noetherian right bounded prime ring such that every proper homomorphic image is an ipri-ring. Then $R$ is a Dedekind prime ring.*

PROOF. Let $I$ be a nonzero ideal of $R$. There exist a positive integer $k$ and prime ideals $P_1, \ldots, P_k$ such that $P_1 \cdots P_k \subseteq I$. By Lemma 9 the ring $R/I$ is right Artinian. In addition $R/I$ is an ipri-ring and hence $R/I$ is a principal right ideal ring by [6, Theorem 2.8]. Thus every proper homomorphic image of $R$ is a principal right ideal ring and every nonzero prime ideal is maximal. By Theorem 6, $R$ is right hereditary. Since $R$ is left Noetherian it follows that $R$ is left hereditary and by combining Lemmas 2 and 3 with [1, Theorem 1.2] we see that $R$ is a Dedekind prime ring.

LEMMA 11. *Let $R$ be a Noetherian prime ring such that the ring $R/PQ$ is an ipri-ring for all nonzero prime ideals $P$, $Q$. Then the ring $R/I$ is an ipri-ring for all nonzero ideals $I$ of $R$.*

PROOF. Note first that nonzero prime ideals of $R$ are maximal ideals. Let $P \subsetneq M$ be distinct prime ideals of $R$ with $P \neq 0$. Then $R/P^2$ is an ipri-ring. Let $\bar{R} = R/P^2$ and $^-$ denote images in $\bar{R}$. Then $\bar{P} = \overline{MP}$ by the proof of Lemma 3 (ii) $\Rightarrow$ (i). Thus $\bar{P} \subseteq \bigcap_{n=1}^{\infty} \bar{M}^n$ and by Lemma 8 there exists an element $m$ in $\bar{M}$ such that $(1 - \bar{m})\bar{P} = 0$. By [6, Corollary 3.4], $1 - \bar{m}$ is nilpotent and hence $M = R$.

Now let $P$ and $Q$ be distinct maximal ideals of $R$. By hypothesis the ring $R^* = R/PQ$ is an ipri-ring. If $*$ denotes images in $R^*$ then $P^* = p^*R^*$ for some element $p$ in $R$. Since $P \not\subseteq Q$, it follows that $p \in \mathcal{C}(Q)$ (see [2, Theorems 3.9]). Hence $P^* \cap Q^* \subseteq p^*Q^* = 0$. Thus $P \cap Q \subseteq PQ$ and this gives $P \cap Q = PQ$. Similarly by considering the ring $R/QP$ we have $P \cap Q = QP$. Thus nonzero prime ideals of $R$ commute.

Let $I$ be a nonzero ideal of $R$. Then there exist positive integers $k, n_1, \ldots, n_k$ and distinct maximal ideals $M_1, \ldots, M_k$ such that

$$M_1^{n_1} M_2^{n_2} \cdots M_k^{n_k} \subseteq I.$$

By the Chinese Remainder Theorem

$$R/(M_1^{n_1} \cdots M_k^{n_k}) \cong \bigoplus_{i=1}^{k} (R/M_i^{n_i}).$$

Moreover, by [5, Theorem 4.1] the ring $R/M_i^{n_i}$ is an ipri-ring for each $1 < i < k$. It follows that the ring $R/(M_1^{n_1} \cdots M_k^{n_k})$, and hence the ring $R/I$, is an ipri-ring.

Combining Lemma 11 with Theorems 6 and 10 we have as our final result

COROLLARY 12. *Let $R$ be a right Noetherian right bounded prime ring such that either* (a) *nonzero prime ideals of $R$ are maximal and $R/PQ$ is a principal right ideal ring for all nonzero prime ideals $P$ and $Q$ of $R$, or* (b) *$R$ is left Noetherian and $R/PQ$ is an ipri-ring for all nonzero prime ideals $P$ and $Q$ of $R$. Then $R$ is right hereditary.*

[9] gives an example of a Noetherian simple ring $S$ which has infinite global dimension but Krull dimension one. It is shown in [8] that if $E$ is an essential right ideal of $S$ and $c$ any regular element in $E$ then the right $S$-module $E/cS$ is cyclic. This example highlights the fact that Theorem 6 is a result about right bounded rings.

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Department of Mathematics, University of Glasgow, Glasgow G12 8QW, Scotland