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TWO CONJECTURES IN THE THEORY OF POINCARÉ DUALITY GROUPS

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ABSTRACT. We show that it is not possible both to realise every Poincaré Duality group as an aspherical manifold and to construct, for each Poincaré complex X, a Poincaré Duality group having the same integral homology as X.

In this note we wish to point out a relationship between two problems in the theory of Poincaré Duality groups, which, for dialectical purposes, we state as conjectures. They are

Homology Equivalence Conjecture. For any finite Poincaré complex X, not homotopy equivalent to S^2 or $\mathbf{R}P^2$, there is a finitely presented Poincaré Duality group G and a homology equivalence f: $K(G, 1) \to X$.

Realisation Conjecture. If G is a finitely presented Poincaré Duality group then there is a closed manifold X_G of homotopy type $K(G, 1)$.

By a homology equivalence f: $Y \to X$ we mean a map which induces homology/cohomology isomorphisms with respect to all local coefficient systems on X. The Homology Equivalence Conjecture is the natural extension to Poincaré complexes of a problem first raised for smooth closed manifolds by Kan and Thurston in [5]. We shall prove

THEOREM 1. *The Homology Equivalence and Realisation Conjectures are not both true.*

We begin with

PROPOSITION 2. *If f: $X \to Y$ is a homology equivalence between connected spaces X, Y which have the homotopy type of* CW *complexes, then, for any loopspace B, f^*: $[Y, B] \to [X, B]$ is bijective.*

PROOF. The suspension of f, Sf: $SX \to SY$, induces isomorphisms in singular homology with simple integer coefficients, so, since SX and SY are simply connected, is a homology equivalence. Hence, for any space Z, we have bijections

$$(Sf)^*: [SY, Z] \xrightarrow{\simeq} [SX, Z] \quad \text{and} \quad f^*: [Y, \Omega Z] \xrightarrow{\simeq} [X, \Omega Z].$$

The results follows on writing $B = \Omega Z$. Q.E.D.

Recall that to any finitely dominated Poincaré complex, more generally to any space X satisfying Poincaré Duality with integer coefficients (possibly twisted) we can associate a stable spherical fibration ν_X, the Spivak fibration of X [3], [7], which

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plays the role of a generalised normal bundle. There is the following, first proved in this generality by Spivak, generalising an earlier result of Atiyah [**1**], [**7**].

PROPOSITION 3 (ATIYAH-SPIVAK). *Let $f\colon X \to Y$ be a homology equivalence between finite orientable Poincaré complexes. Then $f^*(\nu_Y) = \nu_X$.*

PROOF OF THEOREM 1. We shall show that if the Homology Equivalence Conjecture is true then the Realisation Conjecture is false.

Let BG (resp. $BTOP$) be the classifying space for stable spherical fibrations (resp. stable topological microbundles). BG and $BTOP$ are both loopspaces; in fact, they are infinite loopspaces [**2**]. Choose a simply connected Poincaré complex X such that, if ν_X is classified by $c_X \in [X, BG]$, then c_X does not belong to $\operatorname{Im} J$, $J\colon [X, BTOP] \to [X, BG]$. For example, as X we may take the 5-dimensional Poincaré complex with $e_1(X) \neq 0$ constructed by Gitler and Stasheff in [**4**]. As mentioned in [**4**], the universal class e_1 was first introduced as the first obstruction to constructing a cross-section of the fibration $BPL \to BG$, $e_1 \in H^3(BG; \mathbf{Z}_2) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. However, as is now well known, e_1 is the first obstruction to sectioning $BTOP \to BG$. This follows easily from the homotopy equivalence $TOP/PL \simeq K(\mathbf{Z}_2, 3)$ of Kirby-Siebenmann [**6**]. Hence c_X does not belong to the image of $J\colon [X, BTOP] \to [X, BG]$.

Now assume that the Homology Equivalence Conjecture is true, and apply it to produce a finitely presented Poincaré Duality group H and a homology equivalence $f\colon K(H, 1) \to X$. If $K(H, 1)$ is not homotopy equivalent to a finite complex then, by a result of Kirby-Siebenmann [**6**], $K(H, 1)$ is not homotopy equivalent to a closed manifold, disproving the Realisation Conjecture. Hence suppose that $K(H, 1)$ is equivalent to a finite complex. By Proposition 2, we have a commutative square

$$\begin{array}{ccc} [X, BTOP] & \overset{f^*}{\underset{\cong}{\to}} & [K(H, 1), BTOP] \\ \downarrow J & & \downarrow J \\ [X, BG] & \overset{f^*}{\underset{\cong}{\to}} & [K(H, 1), BG] \end{array}$$

Now H is certainly orientable, since $H_1(H, \mathbf{Z}) \cong H_1(X, \mathbf{Z}) = 0$, so that, by Proposition 3, $f^*(c_X) = c_{K(H,1)}$. Since c_X does not lift to $BTOP$, neither does $c_{K(H,1)}$, and $K(H, 1)$ is not homotopy equivalent to any closed topological manifold. Q.E.D.

Of course, the possibility remains that both conjectures are false.

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